VANDERBILT UNIVERSITY

MATH 2610 – ORDINARY DIFFERENTIAL EQUATIONS

Examples of section 12.6

Question 1. Show that the equation

$$\ddot{x} + (x^4 + \dot{x}^2 - 1)\dot{x} + x = 0$$

admits a non-constant periodic solution.

Solution 1. Set $y = \dot{x}$ so that the equation becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -(x^4 + y^2 - 1)y - x. \end{aligned}$$

We will apply the Poincaré-Bendixson theorem. Start noting that $(0, 0)$ is the only critical point of the system. Consider the function $V(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$. Then

$$\begin{aligned} \frac{d}{dt}V(x(t), y(t)) &= x\dot{x} + y\dot{y} \\ &= xy + y(-(x^4 + y^2 - 1)y - x) \\ &= (1 - (x^4 + y^2))y^2. \end{aligned}$$

Consider the curve γ given by $x^4 + y^2 = 1$. Then, $\frac{d}{dt}V(x(t), y(t))$ is ≥ 0 inside γ and ≤ 0 outside γ. The curve γ lies between the circle $x^2 + y^2 = 1$ and the square $\{(x, y) \in \mathbb{R}^2 \mid \max(|x|, |y|) = 1\}$, touching them at the points $(\pm 1, 0)$ and $(0, \pm 1)$. We can choose as the region R of the Poincaré-Bendixson theorem any annulus $r_A \leq x^2 + y^2 \leq r_B$ with $0 < r_A < 1$ and $r_B > \sqrt{2}$.