VANDERBILT UNIVERSITY

MATH 2610 – ORDINARY DIFFERENTIAL EQUATIONS

Examples of section 12.3

Question 1. Show that the given system is almost linear near the origin and discuss the type and stability of the critical point at the origin.

$$\begin{cases} x' = -2x + 2xy, \\ y' = x - y + x^2. \end{cases}$$

Solution 1. The system has the form

$$\begin{cases} x' = ax + by + F(x,y), \\ y' = cx + dy + G(x,y), \end{cases}$$

with $a = -2$, $b = 0$, $c = 1$, $d = -1$, $F(x,y) = 2xy$, and $G(x,y) = x^2$. Thus, $ad - bc \neq 0$ (recall that, by definition, an almost linear system must satisfy $ad - bc \neq 0$). We need to check

$$\frac{F(x,y)}{\| (x,y) \|} = \frac{F(x,y)}{\sqrt{x^2+y^2}} \to 0,$$

and

$$\frac{G(x,y)}{\| (x,y) \|} = \frac{G(x,y)}{\sqrt{x^2+y^2}} \to 0,$$

as $\| (x,y) \| \to 0$. For $x > 0$ and $y > 0$ we have

$$\begin{aligned} \frac{F(x,y)}{\sqrt{x^2+y^2}} &= \frac{2xy}{\sqrt{x^2+y^2}} \\ &= \frac{2xy}{\sqrt{x^2+y^2}} \\ &= \frac{2}{\sqrt{x^2+y^2}/\sqrt{x^2y^2}} \\ &= \frac{2}{\sqrt{\frac{1}{y^2}+\frac{1}{x^2}}} \end{aligned}$$

so that $F(x,y) \to 0$ when $\| (x,y) \| \to 0$, with a similar argument when x, y, or both, are negative. For $G(x,y)$, notice that

$$0 \leq \frac{G(x,y)}{\sqrt{x^2+y^2}} = \frac{x^2}{\sqrt{x^2+y^2}} \leq \frac{x^2}{\sqrt{x^2}}.$$

so $G(x,y) \to 0$ when $\| (x,y) \| \to 0$ by the squeeze theorem. Therefore the system is almost linear.

The eigenvalues of the associated linear system are -1 and -2, and we conclude that $(0,0)$ is an asymptotically stable improper node.