MATH 2610 - TEST 2 SOLUTIONS

VANDERBILT UNIVERSITY

NAME:

Directions:
- Unless stated otherwise, the notation and conventions used in class apply to this test.
- Provide full justifications for your answer. Answers without justification will receive little or no credit.
- Write clearly and legibly.

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
TOTAL	100	

List of formulas

Below are formulas you are allowed to use. Note that it is not said for which kind of equation or in which context each formula applies. You need to recognize them from class and the homework.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$x_p(t) = X(t)\int (X(t))^{-1} f(t)\, dt,$$

Question 1. (20 points). Give the form of the particular solution for the systems below. You do not have to find the constants involved in the particular solution. (*Hint:* There are almost no computations to be done in this question.)

(a) (6 points) $x' = Ax + f$

where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 4 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } f(t) = \begin{bmatrix} 1+t \\ t^3 \\ 0 \end{bmatrix}.$$

(b) (7 points) $x' = Ax + f$

where

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \text{ and } f(t) = \begin{bmatrix} e^t \\ 1 \end{bmatrix}.$$

Hint: $(3, 1)$ and $(1, 1)$ are eigenvectors of A.

(c) (7 points) $x' = Ax + f$

where A is a 3×3 matrix with eigenvalues $1 \pm i$ and -1 and

$$f(t) = e^t \begin{bmatrix} t \\ 2 \\ 1 \end{bmatrix}.$$

Solution 1. (a) A is a lower-triangular matrix, thus its eigenvalues are the elements in the diagonal. Since they are distinct, A admits three linearly independent eigenvectors u_1, u_2, and u_3. Thus, $x_1(t) = e^{-t}u_1$, $x_2(t) = e^{2t}u_2$, and $x_3(t) = e^t u_3$ are three linearly independent solutions of $x' = Ax$. Since f is a polynomial in t and x_1, x_2, and x_3 are not polynomials, the particular solution will have the same form as f. Therefore

$$x_p(t) = t^3 a + t^2 b + tc + d,$$

where a, b, c, and d are constant vectors. Note that we do not need to find u_1, u_2, or u_3 to answer this question.

(b) To say that $u_1 = (3, 1)$ and $u_2 = (1, 1)$ are eigenvectors means that $Au_1 = \lambda_1 u_1$ and $Au_2 = \lambda_2 u_2$. Thus

$$\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and we conclude that 1 and -1 are eigenvalues of A. The vectors $(3, 1)$ and $(1, 1)$ are clearly linearly independent (we can also conclude this from the fact that the corresponding eigenvectors are distinct). It follows that $x_1(t) = e^t(3, 1)$ and $x_2(t) = e^{-t}(1, 1)$ are two linearly independent solutions of $x' = Ax$. The non-homogeneous term can be written as $e^t(1, 0) + (0, 1)$, and we see that it repeats the form of x_1. Therefore, the form of the particular solution is

$$x_p(t) = te^t a + e^t b + c,$$

where a, b, and c are constant vectors.

(c) There are two linearly independent solutions x_1 and x_2 to $x' = Ax$ which are associated with the complex conjugate eigenvalues; they are combinations of $e^t \cos t$ and $e^t \sin t$. The third eigenvalue -1 is real, hence distinct from the previous two, and therefore a third linearly independent solution to the associated homogeneous equation is proportional to e^{-t}. Because f does not repeat the form of any of such solutions, we have

$$x_p(t) = e^t(at + b),$$

where a and b are a constant vectors.

Question 2. (20 points) Consider the system

$$x' = Ax + f, \tag{1}$$

where A is a $n \times n$ matrix function and f is a vector function.

(a) (5 points) State the variation of parameters formula for particular solutions of (1).

(b) (5 points) Give conditions on A and f guaranteeing that the formula you wrote in (a) is well-defined.

(c) (5 points) Prove the formula you stated in (a).

(d) (5 points) State conditions on A and f that guarantee that the initial value problem

$$\begin{aligned} x' &= Ax + f, \\ x(t_0) &= x_0, \end{aligned} \tag{2}$$

admits a unique solutions. Then, based on the formula you stated in (a), find a formula for the solution of (2)

Solution 2. (a) The formula is

$$x_p(t) = X(t) \int (X(t))^{-1} f(t)\, dt,$$

where X is a fundamental matrix for the associated homogeneous equation.

(b) The integral has to be well-defined (X^{-1} will always exist because by assumption X is a fundamental matrix). A sufficient condition for the integral to make sense is that X^{-1} and f be continuous.

(c) Let X be as above. Set $x_p = Xv$, where v a vector function to be determined. Plugging in:

$$x_p' = (Xv)' = X'v + Xv' = Ax_p + f = AXv + f.$$

Recalling that $X' = AX$, we conclude $Xv' = f$. Solving for v', integrating (without adding a constant of integration) and plugging v back into x_p, yields the result.

(d) A sufficient condition for the existence of a unique solution on an interval containing t_0 is that A and f be continuous. Such a unique solution can be written as

$$x = Xc + x_p,$$

where c is a constant vector determined by the initial conditions. Using (a), we can write

$$x(t) = X(t)c + X(t) \int_{t_0}^{t} (X(s))^{-1} f(s)\, ds,$$

from which we find $c = X(t_0)^{-1} x_0$, thus

$$x(t) = X(t)X(t_0)^{-1}x_0 + X(t) \int_{t_0}^{t} (X(s))^{-1} f(s)\, ds.$$

Question 3. (20 points) Consider the system $x' = Ax$, where

$$A = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}.$$

To answer the questions that follow, you are allowed to use the following facts:

(i) The matrix A has eigenvalue -1 with multiplicity two.

(ii) A possesses only one linearly independent eigenvector, which can be taken to be $(1, 2)$.

(iii) We have

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(a) (10 points) Find a fundamental matrix for the given system.

(b) (10 points) Using the fundamental matrix you found in (a), compute e^{At}.

Solution 3. Because A has only one eigenvalue, -1, and only one linearly independent corresponding eigenvector, the eigenvectors of A produce only one solution given by eigenvectors, which we can take to be $x_1(t) = e^{-t}(1, 2)$. To find a second linearly independent solution, we need to calculate the generalized eigenvectors of A. Compute

$$A - \lambda I = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$

and we need to solve $(A - \lambda I)^2 u = 0$, i.e.,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Any u will be a solution, but since we want a generalized eigenvector, we need to choose u non-zero, and because we want a second linearly independent solution, u needs to be linearly independent from $(1, 2)$. We can take, e.g., $(1, 0)$. Then

$$\begin{aligned} x_2(t) &= e^{-t}(u + t(A - (-1)I)u) \\ &= e^{-t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\ &= e^{-t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) \\ &= e^{-t}\begin{bmatrix} 1 + 2t \\ 4t \end{bmatrix}. \end{aligned}$$

Therefore, a fundamental matrix is

$$X(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} = e^{-t}\begin{bmatrix} 1 & 1 + 2t \\ 2 & 4t \end{bmatrix}$$

(b) We have $e^{At} = X(t)(X(0))^{-1}$, thus

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & 1 + 2t \\ 2 & 4t \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} = -\frac{1}{2}e^{-t}\begin{bmatrix} 1 & 1 + 2t \\ 2 & 4t \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + 2t & -t \\ 4t & 1 - 2t \end{bmatrix}. \end{aligned}$$

Question 4. (20 pts) True or false? Justify your answer.

(a) (5 points) Let $x_1, \cdots, x_n$ be n-component vector functions defined on an interval I such that $W[x_1, \dots, x_n]$ does not vanish identically on I, where $W[x_1, \dots, x_n]$ is the Wronskian of $x_1, \dots, x_n$. Then $x_1, \dots, x_n$ are linearly independent.

(b) (5 points) Let $x_1, \cdots, x_n$ be n-component vector functions defined on an interval I such that $W[x_1, \dots, x_n]$ vanishes identically on I, where $W[x_1, \dots, x_n]$ is the Wronskian of $x_1, \dots, x_n$. Then $x_1, \dots, x_n$ are linearly dependent.

(c) (5 points) Let A be a $n \times n$ matrix and u a generalized eigenvector of A associated to an eigenvalue λ. Then $e^{\lambda t}u$ is a solution to $x' = Ax$.

(d) (5 points) If A and B are $n \times n$ matrices, then $e^{A+B} = e^A e^B$.

Solution 4. (a) True, this is precisely the statement of one of the theorems proven in class.

(b) False. The functions $(t, |t|)$ and $(|t|, t)$ are linearly independent on $(-\infty, \infty)$ but their Wronskian vanishes.

(c) False. For example, taking the generalized vector $(1, 0)$ from question 3, we see that

$$(e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix})' = -e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix} e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(d) False. This is true if A and B commute.

Question 5. (20 points) Let A be a $n \times n$ matrix.

(a) (6 points) Define $\|A\|$ and explain why it is well-defined (you do not need to prove that it is in fact a norm, only explain why it is well-defined).

(b) (7 points) State the definition of e^A.

(c) (7 points) Prove that e^{At} is a fundamental matrix for the system $x' = Ax$.

Solution 5. (a) The definition is

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$

where the norms on the right-hand side are the standard norm in $\mathbb{R}^n$. This is well-defined because the map $x \mapsto Ax$ is continuous, the norm in $\mathbb{R}^n$ is a continuous function, and the set

$$S = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

is compact, thus by the extreme value theorem $\|Ax\|$ achieves a maximum over S.

(b) The definition is

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!},$$

where $A^0 = I$.

(c) We have

$$\frac{d}{dt} e^{At} = \frac{d}{dt} \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0}^{\infty} k \frac{A^k t^{k-1}}{k!} = A \sum_{k=1}^{\infty} \frac{A^{k-1} t^{k-1}}{(k-1)!} = A \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = Ae^{At},$$

thus e^{At} satisfies the matrix differential equation $X' = AX$. Because e^{At} is invertible, e^{At} is a fundamental matrix.