MATH 2610 - TEST 1

VANDERBILT UNIVERSITY

NAME:

Directions:

- Unless stated otherwise, the notation and conventions used in class apply to this test.
- Provide full justifications for your answer. Answers without justification will receive little or no credit.
- Write clearly and legibly.

Question	Points	Score
1	15	
2	20	
3	20	
4	20	
5	25	
TOTAL	100	

List of formulas

Below are formulas you are allowed to use. Note that it is not said for which kind of equation or in which context each formula applies. You need to recognize them from class and the homework.

$$x(t) = e^{-\int P(t)\,dt}\left(\int e^{\int P(t)\,dt} Q(t)\,dt + C\right).$$

$$W(x_1, x_2) = x_1 x_2' - x_1' x_2.$$

$$x_p(t) = -x_1(t)\int \frac{f(t)x_2(t)}{a(t)W(x_1,x_2)(t)}\,dt + x_2(t)\int \frac{f(t)x_1(t)}{a(t)W(x_1,x_2)(t)}\,dt.$$

$$x_2(t) = x_1(t)\int \frac{e^{-\int p(t)\,dt}}{(x_1(t))^2}\,dt.$$

Question 1. [15 points] For each differential equation below, state its order and whether it is linear or nonlinear.

(a) [3 points] $x' + x^2 = 0.$

(b) [3 points] $y'' + (1+t)y' + y = e^t.$

(c) [4 points] $y'' + (1+t)y' + y = e^y.$

(d) [5 points] $\dfrac{d^3y}{dx^3} - \dfrac{d^4}{dx^4}\sin^2(3x) + x = 0.$

Solution 1. (a) First-order, nonlinear. (b) Second-order, linear. (c) Second-order, nonlinear. (d) Third-order, linear.

Question 2. **[20 points]** Find the general solution of the following differential equations.

(a) **[6 points]** $y' - t(y-1) = 0$.

(b) **[7 points]** $(1+x^2)y' + xy - x = 0$.

(c) **[7 points]** $(4y + 3x^2 - 3xy^2)y' - y^3 + 6xy = 0$.

Solution 2. (a) This is a separable equation. $y = 1$ is a solution. For $y \neq 1$, write $\frac{dt}{y-1} = t\,dt$. Integrating, $\ln|y-1| = \frac{1}{2}t^2 + C$, or $|y-1| = Ce^{\frac{1}{2}t^2}$, or yet $y = Ce^{\frac{1}{2}t^2} + 1$. Since $C = 0$ gives $y = 1$, this is the general solution.

(b) This is a linear equation with $P(x) = Q(x) = \frac{x}{x^2+1}$. Compute $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(1+x^2) = \ln\sqrt{1+x^2}$. Then

$$\int e^{\int P(x)\,dx} Q(x)\,dx = \int \frac{x}{\sqrt{1+x^2}}\,dx = \sqrt{1+x^2}.$$

The formula for solutions to first-order linear equations gives

$$y(x) = 1 + \frac{C}{\sqrt{1+x^2}}.$$

(c) (This equation was worked out in class as an example.) Write the equation as

$$\underbrace{(6xy - y^3)}_{=M(x,y)}\,dx + \underbrace{(4y + 3x^2 - 3xy^2)}_{N(x,y)}\,dy = 0.$$

Check $\frac{\partial M}{\partial y} = 6x - 3y^2 = \frac{\partial N}{\partial x}$, so the equation is exact. Set

$$F(x,y) = \int M(x,y)\,dx = \int 6xy - y^3\,dx = 3x^2y - xy^3 + g(y).$$

Compute

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(3x^2y - xy^3 + g(y)) = N = 4y + 3x^2 - 3xy^2,$$

giving $g'(t) = 4y$ thus $g(y) = 2y^2$. The general solution is

$$3x^2y - xy^3 + 2y^2 = C.$$

Question 3. [20 points] Find the form of the particular solution for the differential equations below. You do *not* have to find the value of the constants in the particular solution.

(a) [6 points] $x'' - 3x' + 2x = t^3$.

(b) [7 points] $x'' + 16x = e^t \sin t$.

(c) [7 points] $x'' - 4x' + 4x = (t^2 + t)e^{2t}$.

Solution 3. (a) The characteristic equations for the AHE is $\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0$, giving $x_1 = e^t$, $x_2 = e^{2t}$. The general form for the inhomogeneous term does not repeat any term in x_1 or x_2, thus $x_p = At^3 + Bt^2 + Ct + D$.

(b) The characteristic equations for the AHE is $\lambda^2 + 16 = 0$, giving $\lambda = \pm 4i$. Then $x_1 = \cos(4t)$ and $x_2 = \sin(4t)$. The general form for the inhomogeneous term does not repeat any term in x_1 or x_2, thus $x_p = e^t(A \cos t + B \sin t)$.

(c) The characteristic equations for the AHE is $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$, giving $x_1 = e^{2t}$, $x_2 = te^{2t}$. The general form for the inhomogeneous term repeats both x_1 and x_2, thus $x_p = t^2(At^2 + B^t + C)e^{2t}$.

Question 4. [20 points] Find the general solution to the differential equations below. You do *not* have to compute the integrals that appear in the solutions, just leave them indicated.

(a) [10 points] $t^2x'' + 7tx' - 7x = t^2 + 1$, $t > 0$.

(b) [10 points] $tx'' + (1-2t)x' + (t-1)x = e^{2t}$, $t > 0$.

Hint: e^t *is a solution of the associated homogeneous equation.*

Solution 4. (a) This is a Cauchy-Euler equation with characteristic equation $\lambda^2 + 6\lambda - 7 = 0$. The roots are $\lambda_1 = 1$ and $\lambda_2 = -7$, giving $x_1 = t$, $x_2 = t^{-7}$. The Wronskian is

$$W(x_1, x_2) = t(t^{-7})' - t't^{-7} = -8t^{-7}.$$

Variation of parameters gives

$$\begin{aligned} x_p(t) &= -x_1(t) \int \frac{f(t)x_2(t)}{a(t)W(x_1,x_2)(t)}\,dt + x_2(t) \int \frac{f(t)x_1(t)}{a(t)W(x_1,x_2)(t)}\,dt \\ &= -t \int \frac{(t^2+1)t^{-7}}{t^2(-8t^{-7})}\,dt + t^{-7} \int \frac{(t^2+1)t}{t^2(-8t^{-7})}\,dt \\ &= \frac{1}{8}t \int \frac{t^2+1}{t^2}\,dt - \frac{1}{8}t^{-7} \int (t^2+1)t^6\,dt \end{aligned}$$

and $x = c_1x_1 + c_2x_2 + x_p$.

(b) The formula for a second linearly independent solution gives

$$\begin{aligned} x_2(t) &= x_1(t) \int \frac{e^{-\int p(t)\,dt}}{(x_1(t))^2}\,dt \\ &= e^t \int \frac{e^{-\int \frac{1-2t}{t}\,dt}}{e^{2t}}\,dt \\ &= e^t \int \frac{e^{-\ln t + 2t}}{e^{2t}}\,dt \\ &= e^t \ln t. \end{aligned}$$

The Wronskian becomes

$$W = e^t(e^t \ln t)' - (e^t)'e^t \ln t = \frac{e^{2t}}{t}.$$

Variation of parameters gives

$$\begin{aligned} x_p(t) &= -x_1(t) \int \frac{f(t)x_2(t)}{a(t)W(x_1,x_2)(t)}\,dt + x_2(t) \int \frac{f(t)x_1(t)}{a(t)W(x_1,x_2)(t)}\,dt \\ &= -e^t \int \frac{e^{2t}e^t \ln t}{t\frac{e^{2t}}{t}}\,dt + e^t \ln t \int \frac{e^{2t}e^t}{t\frac{e^{2t}}{t}}\,dt, \\ &= -e^t \int e^t \ln t\,dt + e^t \ln t \int e^t\,dt, \end{aligned}$$

and $x = c_1x_1 + c_2x_2 + x_p$.

Question 5. [25 points]
(a) [8 points] State a theorem on existence and uniqueness of solutions to the initial-value problem

$$\frac{dy}{dx} = f(x, y), \tag{DE}$$
$$y(x_0) = y_0. \tag{IC}$$

(b) [7 points] Show that the theorem you stated in part (a) is applicable if $f(x, y) = e^{y^2} - 1$.

(c) [10 points] Let $Y = Y(x)$ be a solution to the initial-value problem (DE)-(IC) with f as in part (b) and $y_0 \neq 0$. Does there exist a x_* such that $Y(x_*) = 0$? In other words, can the graph of the solution Y touch the x-axis?
Hint: First, note that $y(x) = 0$ solves the differential equation (DE) with f as in part (b). Suppose that $Y(x_*) = 0$ for some x_*. Apply the theorem you stated in (a).

Solution 5. (a) Assume that f and $\partial_y f$ are continuous on an open rectangle containing the point (x_0, y_0). Then (DE)-(IC) has a unique solution defined on an open interval containing the point x_0.

(b) $\partial_y(e^{y^2} - 1) = 2ye^{y^2}$ is always continuous.

(c) Suppose that $Y(x_*) = 0$. Apply the theorem from part (a) with x_0 replaced by x_* and y_0 replaced by 0. Then Y is a solution to this IVP, but so is $y(x) = 0$. Since Y is not the zero function (because it solved an IVP with $y_0 \neq 0$), we would obtain two different solutions, contradicting uniqueness. Thus, Y cannot touch the x-axis.