MATH 2610 - PRACTICE FOR TEST 3

VANDERBILT UNIVERSITY

Question 1. The questions that follow refer to the system

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y).$$

(a) What is a critical point for this system and how is it related to solutions of the system?

(b) What is an isolated critical point?

(c) Define what it means to say that a critical point is stable, asymptotically stable, and unstable. Illustrate the definitions with pictures.

(d) Define an almost linear system near the origin.

Question 2. Consider the linear system

$$\dot{x} = ax + by,$$
$$\dot{y} = cx + dy,$$

and suppose that $ad - bc \neq 0$. Let λ_1 and λ_2 be the eigenvalues of the system. Based on the definition of stability/instability you gave in question 1, show that:

(a) The system is asymptotically stable if $\lambda_1, \lambda_2 < 0$.

(b) The system is unstable if one of the eigenvalues is positive.

Question 3. Show that the system

$$\dot{x} = e^{x+y} - \cos x,$$
$$\dot{y} = \cos y + x - 1,$$

is almost linear near the origin and discuss its stability.

Question 4. Consider the DE

$$\ddot{x} + e^x - 1 = 0.$$

(a) Explain why this is a conservative system.

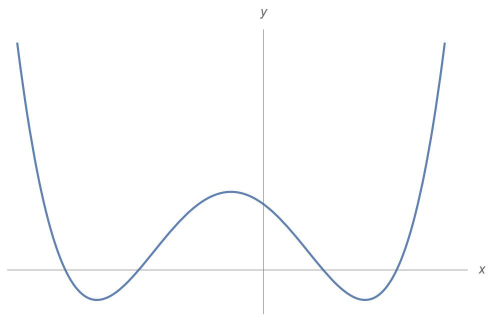
(b) Find the potential function G.

(c) Find the energy function $E(x, v)$. Select it so that $E(0, 0) = 0$.

(d) Write the DE as a first order system and determine its critical points.

(e) Determine the stability of the critical points.

Question 5. Consider a conservative system whose potential function is given by the graph below. Sketch the phase portrait of the system.



Question 6. Consider the system

$$\dot{x} = x^3y + x^2y^3 - x^5,$$
$$\dot{y} = -2x^4 - 6x^3y^2 - 2y^5.$$

(a) Show that the origin is a critical point.

(b) Explain why this system is not almost linear.

(c) Determine the stability of the origin. To do so, you can use, without proving it, that the origin is an isolated critical point. *Hint:* the function $ax^2 + by^2$ is useful.

Question 7. Determine the stability of the origin for the system

$$\dot{x} = 2x^3,$$
$$\dot{y} = 2x^2y - y^3.$$

Hint: the function $x^2 - y^2$ is useful.

Question 8. Prove that the equation

$$\ddot{x} + (x^4 + (\dot{x})^2 - 1)\dot{x} + x = 0$$

has a non-constant periodic solution.