VANDERBILT UNIVERSITY

MATH 2420 – METHODS OF ORDINARY DIFFERENTIAL EQUATIONS

Practice for test 1 — Solutions

Directions: This practice test should be used as a study guide, illustrating the concepts that will be emphasized in the test. This does not mean that the actual test will be restricted to the content of the practice. Try to identify, from the questions below, the concepts and methods that you should master for the test. For each question in the practice test, study the ideas and techniques connected to the problem, even if they are not directly used in your solution.

Take this also as an opportunity to practice how you will write your solutions in the test. For this, write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

The first test will cover all material discussed up to (including) section 4.7.

Question 1. For each equation below, identify the unknown function, classify the equation as linear or non-linear, and state its order.

(a) $\sqrt{y}\dfrac{dy}{dx} + x^2 y = 0$.

(b) $u'' + u = \cos x$

(c) $x''' = -\sin x\, x'$.

Solution 1. (a) Unknown: y. Non-linear. First order. (b) Unknown: u. Linear. Second order. (c) Unknown: x. Non-linear. Third order.

Question 2. Find the general solution of the given differential equation.

(a) $x' - x^2 = 0$.

(b) $x'' + 5x' + 6x = e^{2t}$.

(c) $y' = -\dfrac{4x^3 + y}{4y^3 + x}$.

Solution 2. (a) This is a separable equation. For $x \neq 0$

$$\frac{dx}{x^2} = dt \Rightarrow \frac{1}{x} = -t + C \Rightarrow x = \frac{1}{C - t},$$

where C is an arbitrary constant. We immediately verify that $x = 0$ is also a solution, hence the general solution is $x = 1/(C - t)$ or $x = 0$.

(b) This is a linear second order inhomogeneous equation. The characteristic equation is $\lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3) = 0$. Hence $x_h = c_1 e^{-2t} + c_2 e^{-3t}$, where c_1 and c_2 are arbitrary constants, is the general solution of the associated homogeneous equation. Since this does not contain e^{2t}, we seek a particular solution in the form $x_p = Ae^{2t}$, A constant. Plugging in produces

$$4A + 10A + 6A = 1 \Rightarrow A = \frac{1}{20}.$$

The general solution is $x = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{20} e^{2t}$.

(c) Write the equation as

$$(4x^3 + y)\,dx + (4y^3 + x)\,dy = M\,dx + N\,dy = 0.$$

Then, $\partial_y M = 1 = \partial_x N$ and this is an exact equation. Set

$$F(x, y) = \int (4x^3 + y)\,dx = x^4 + xy + g(y).$$

Next,

$$\partial_y F = x + g'(y) = N = x + 4y^3 \Rightarrow g'(t) = 4y^3 \Rightarrow g(y) = y^4.$$

The general solution is thus

$$x^4 + xy + y^4 = C,$$

where C is an arbitrary constant.

Question 3. Give the form of the particular solution for the given differential equations. You do not have to find the values of the constants of the particular solution.

(a) $x'' - 3x' + 2x = e^{2t}$.

(b) $x'' + 9x = \sin t$.

(c) $x'' - x = 3t^2 + 1$.

(d) $x'' + x' - 2x = e^{-2t} + e^t$.

Solution 3. (a) The characteristic equation is $(\lambda - 1)(\lambda - 2) = 0$, so e^t and e^{2t} are two linearly independent solutions of the associated homogeneous equation. Since the inhomogeneous term repeats one of these solutions, we have

$$x_p = Ate^{2t},$$

where A is a constant.

(b) The characteristic equation is $\lambda^2 + 9 = 0$, so $\cos(3t)$ and $\sin(3t)$ are two linearly independent solutions of the associated homogeneous equation. Since the inhomogeneous term does not repeat either of these solutions, we have

$$x_p = A\cos t + B\sin t,$$

where A and B are constants.

(c) The characteristic equation is $\lambda^2 - 1 = 0$, so e^t and e^{-t} are two linearly independent solutions of the associated homogeneous equation. Since the inhomogeneous term does not repeat either of these solutions, we have

$$x_p = At^2 + Bt + C,$$

where A, B, and C are constants.

(d) The characteristic equation is $(\lambda+2)(\lambda-1)=0$, so e^{-2t} and e^t are two linearly independent solutions of the associated homogeneous equation. Each inhomogeneous term repeats one of these solutions. Thus, we have

$$x_p = Ate^{-2t} + Bte^t,$$

where A and B are constants.

Question 4. Consider the following initial value problem:

$$\begin{aligned} y' - \sqrt{y} - xy^2 &= 0, \\ y(1) &= a. \end{aligned}$$

Determine for which values of a this problem admits a unique solution.

Solution 4. Write the equations as $y' = f(x,y)$, with $f(x,y) = \sqrt{y} + xy^2$. Note that this problem is not defined for $y < 0$.

By the the existence and uniqueness theorem for first order equations seen in class, a solution satisfying $y(x_0) = y_0$ will exist and be unique in a neighborhood of $x = x_0$ if $\partial_y f$ exists and is continuous in a neighborhood of (x_0, y_0).

In our case, $\partial_y f(x,y) = \frac{1}{2}\frac{1}{\sqrt{y}} + 2xy$, which is continuous and well-defined for all x and all $y > 0$. Hence, the given initial value problem admits a unique solution if $a > 0$.

Question 5. Find a particular solution to:

$$x'' + 9x = \sec^2(3t)$$

Solution 5. Two linearly independent solutions of the associated homogeneous equation are $\cos(3t)$ and $\sin(3t)$. Using the formula

$$x_p(t) = -x_1(t)\int \frac{x_2(t)f(t)}{W(x_1,x_2)(t)}\,dt + x_2(t)\int \frac{x_1(t)f(t)}{W(x_1,x_2)(t)}\,dt,$$

and the fact that $W(\cos(3t), \sin(3t)) = 3$, we find

$$\begin{aligned} x_p(t) &= -\frac{1}{3}\cos(3t)\int \sin(3t)\sec^2(3t)\,dt + \frac{1}{3}\sin(3t)\int \cos(3t)\sec^2(3t)\,dt \\ &= -\frac{1}{9}\cos(3t)\sec(3t) - \frac{1}{9}\sin(3t)\Big(\ln(\cos(\frac{3t}{2}) - \sin(\frac{3t}{2})) - \ln(\cos(\frac{3t}{2}) + \sin(\frac{3t}{2})\Big) \end{aligned}$$

Question 6. Verify that $x_1(t) = t$ is a solution to

$$x'' - \frac{1}{t}x' + \frac{1}{t^2}x = 0,\ t > 0,$$

and find a second linearly independent solution.

Solution 6. The verification is immediate. To find the second linearly independent solution, we use the formula

$$x_2(t) = x_1(t)\int \frac{e^{-\int p(t)\,dt}}{(x_1(t))^2}\,dt$$

with $p(t) = -\frac{1}{t}$ to find

$$x_2(t) = t\int \frac{e^{\int \frac{1}{t}\,dt}}{t^2}\,dt = t\int \frac{1}{t}\,dt = t\ln t.$$

Question 7. True or false? Justify your answers.

(a) If $p(x)$ and $q(x)$ are continuous functions on the interval (a,b), then the initial value problem

$$\begin{aligned} y'(x) + p(x)y(x) &= q(x), \\ y(x_0) &= y_0, \end{aligned}$$

always admits a unique solution for any given $x_0 \in (a,b)$ and $y_0 \in \mathbb{R}$.

(b) Given the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0,$$

it is always possible to find a function $F = F(x,y)$ such that $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$, and the general solution of the differential equation is given by $F(x,y) = C$, where C is an arbitrary constant.

(c) If a, b, and c are constants and $a \neq 0$, the equation

$$ax'' + bx' + cx = 0,$$

always admits two linearly independent solutions $x_1(t)$ and $x_2(t)$ that are defined for all $t \in \mathbb{R}$.

(d) If x_1 and x_2 are two functions such that their Wronskian vanishes, then they are linearly dependent.

Solution 7. (a) True. This follows from the existence and uniqueness theorem for linear first order equations seen in class.

(b) False. According to the study of such equations developed in class, the statement is true if $\partial_y M = \partial_x N$.

(c) True. In class, we established that if λ_1 and λ_2 are the two roots of the characteristic equation, then the two linearly independent solutions are given by $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ if $\lambda_1 \neq \lambda_2$ are real, $e^{\lambda t}$ and $te^{\lambda t}$ if $\lambda_1 = \lambda_2 = \lambda$, and $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$ if $\lambda_1 = \alpha + i\beta$, $\beta \neq 0$.

(d) False. This is true if x_1 and x_2 are also solutions to a second order linear differential equation.

Question 8. State and prove the superposition principle for second order linear differential equations with constant coefficients.

Solution 8. Done in class.

Question 9. Know the statements and how to use the theorems established in class. For theorems that were proved in class, know their proofs.

Solution 9. N/A.

Question 10. Review the homework problems and examples posted in the course webpage.

Solution 10. N/A.