VANDERBILT UNIVERSITY

MATH 2420 – METHODS OF ORDINARY DIFFERENTIAL EQUATIONS

Practice for test 2 – solutions

Directions: This practice test should be used as a study guide, illustrating the concepts that will be emphasized in the test. This does not mean that the actual test will be restricted to the content of the practice. Try to identify, from the questions below, the concepts and methods that you should master for the test. For each question in the practice test, study the ideas and techniques connected to the problem, even if they are not directly used in your solution.

Take this also as an opportunity to practice how you will write your solutions in the test. For this, write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

The first test will cover all material discussed from (including) section 6.1 to (including) section 7.9. (Note that sections 1.3 and 1.4 will not be in the test.)

The table below indicates the Laplace transform $F(s)$ of the given function $f(t)$.

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$

The following are the main properties of the Laplace transform.

Function	Laplace transform
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
$e^{at}f(t)$	$F(s-a)$
$t^nf(t)$	$(-1)^n\frac{d^n}{ds^n}F(s)$
$(f*g)(t)$	$F(s)G(s)$
$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)u(t-a)$	$e^{-as}F(s)$

Above, $f*g$ is the convolution of f and g, given by

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau,$$

and $u(t-a)$ is given by

$$u(t-a)=\begin{cases}0, & t<a,\\ 1, & t>a.\end{cases}$$

Question 1. Solve the following differential equations.

(a) $x''' - 3x'' + 4x = 0$.

(b) $x''' - 3x'' - x' + 3x = 0$

(c) $x'''' + 4x'' + 4x = 0$.

Solution 1. (a) The characteristic equation is $\lambda^3 - 3\lambda^2 + 4 = 0$. Recall the following method for finding a root of a cubic or higher polynomial equation: we try to guess a solution by plugging in numbers that divide the term without λ, in this case $\pm1, \pm2, \pm4$ (if there is no term without λ, then zero is a solution). Plugging in -1 we find that it is a root. Next, we factor the polynomial by $\lambda - \text{root} = \lambda - (-1) = \lambda + 1$. Since factoring a degree one polynomial produces a polynomial of one degree lower (two in this case) we know that

$$(\lambda + 1)(A\lambda^2 + B\lambda + C) = \lambda^3 - 3\lambda^2 + 4.$$

To figure out A, B, and C, we compare both sides. It is easier to star comparing the terms in highest power of λ and without λ. For the former, the only term on the left hand side that produces a λ^3 is $\lambda \times A\lambda^2 = A\lambda^3$, so we must have $A = 1$. For the latter, the only term on the left hand side that produces no λ is $1 \times C = C$, so we must have $C = 4$. Hence

$$(\lambda + 1)(\lambda^2 + B\lambda + 4) = \lambda^3 - 3\lambda^2 + 4.$$

There is no term with λ to the power one on the right hand side, while there are two such terms on the left hand side, $\lambda \times 4 = 4\lambda$ and $1 \times B\lambda = B\lambda$, so we must have $4\lambda + B\lambda = 0$ and thus $B = -4$. Hence $\lambda^3 - 3\lambda^2 + 4 = (\lambda - 4\lambda + 4)(\lambda + 1) = (\lambda - 2)^2(\lambda + 1)$. Using the corresponding form of solutions given in class, we conclude that

$$x(t) = c_1 e^{-t} + c_2 e^{2t} + c_3 t e^{2t}.$$

(b) The characteristic equation is $\lambda^3 - 3\lambda^2 - \lambda + 3 = 0 = (\lambda + 1)(\lambda - 3)(\lambda - 1)$, where we used the method discussed in (a) to factor the polynomial. Thus,

$$x(t) = c_1 e^{-t} + c_2 e^{3t} + c_3 e^{t}.$$

(c) The characteristic equation is $\lambda^4 + 4\lambda^2 + 4$, which we recognize as a perfect square in λ^2: $\lambda^4 + 4\lambda^2 + 4 = (\lambda^2 + 2)^2$. Thus the roots are $\lambda = \pm\sqrt{2}i$ with multiplicity two and we conclude

$$x(t) = c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t) + c_3 t \cos(\sqrt{2}t) + c_4 t \sin(\sqrt{2}t).$$

Question 2. Recall that a function f is said to be of exponential order $\alpha > 0$ if there exist positive constants T and M such that

$$|f(t)| \leq Me^{\alpha t}, \text{ for all } t \geq T.$$

Which of the following functions are of exponential order?

(a) $t \ln t$.

(b) e^{t^3}.

(c) $\dfrac{1}{t^2+1}$.

Solution 2. (a) One method to determine whether a given function $f(t)$ is of exponential order is the following. We compute

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}},$$

with $\alpha > 0$. If the limit is a finite number, then $|\frac{f(t)}{e^{\alpha t}}|$ has to remain bounded for t sufficiently large and, therefore, there must exist constants T and M such that for $t \geq T$ we have

$$\left|\frac{f(t)}{e^{\alpha t}}\right| \leq M \text{ or } |f(t)| \leq Me^{\alpha t}.$$

With $f(t) = t \ln t$ we find that for any $\alpha > 0$

$$\lim_{t\to\infty} \frac{t \ln t}{e^{\alpha t}} = 0,$$

where we used L'Hospital rule to compute the limit. Thus, $t \ln t$ is of exponential order. (To be of exponential order it suffices to find *some* α; thus, we could, for example, have taken $\alpha = 1$ and computed $\lim_{t\to\infty} \frac{t \ln t}{e^t} = 0$.)

(b) From the discussion of part (a) we see that if

$$\lim_{t\to\infty} \left|\frac{f(t)}{e^{\alpha t}}\right| = \infty$$

for any $\alpha > 0$ then $f(t)$ is not of exponential order. In our case

$$\lim_{t\to\infty} \left|\frac{e^{t^3}}{e^{\alpha t}}\right| = \lim_{t\to\infty} e^{t^3-\alpha t} = \infty$$

since for any α we have $\lim_{t\to\infty}(t^3 - \alpha t) = \infty$ and $e^\infty = \infty$, thus e^{t^3} is not of exponential order.

(c) We can proceed as in part (a). However, it is quicker to note that $\frac{1}{t^2+1} \leq 1$ for any t and $e^t \geq 0$ for all $t \geq 0$, thus $\frac{1}{t^2+1} \leq e^t$ for $t \geq 0$ and $\frac{1}{t^2+1}$ is of exponential order.

Question 3. Determine $\mathscr{L}^{-1}\{F\}$. You do not need to determine the constants of the partial fractions.

(a) $F(s) = \dfrac{5s^2+34s+53}{(s+3)^2(s+1)}$.

(b) $s^2F(s) + sF(s) - 6F(s) = \dfrac{s^2+4}{s^2+s}$.

(c) $sF(s) + 2F(s) = \dfrac{10s^2+12s+14}{s^2-2s+2}$.

(d) $F(s) = \ln\left(\dfrac{s^2+9}{s^2+1}\right)$.

Solution 3. (a) The partial fractions reads

$$\frac{5s^2+34s+53}{(s+3)^2(s+1)} = \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{(s+3)^2}$$

so that

$$\begin{aligned}\mathscr{L}^{-1}\{\frac{5s^2+34s+53}{(s+3)^2(s+1)}\} &= A\mathscr{L}^{-1}\{\frac{1}{s+1}\} + B\mathscr{L}^{-1}\{\frac{1}{s+3}\} + C\mathscr{L}^{-1}\{\frac{1}{(s+3)^2}\} \\ &= Ae^{-t} + Be^{-3t} + Cte^{-3t}.\end{aligned}$$

(b) Factoring $F(s)$ and applying partial fractions we have

$$\begin{aligned}F(s) &= \frac{s^2+4}{(s^2+s)(s^2+s-6)} = \frac{s^2+4}{s(s+1)(s-2)(s+3)} \\ &= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-2} + \frac{D}{s+3},\end{aligned}$$

so that

$$\begin{aligned}\mathscr{L}^{-1}\{\frac{s^2+4}{(s^2+s)(s^2+s-6)}\} &= A\mathscr{L}^{-1}\{\frac{1}{s}\} + B\mathscr{L}^{-1}\{\frac{1}{s+1}\} + C\mathscr{L}^{-1}\{\frac{1}{s-2}\} + D\mathscr{L}^{-1}\{\frac{1}{s+3}\} \\ &= A + Be^{-t} + Ce^{2t} + De^{-3t}.\end{aligned}$$

(c) Factoring $F(s)$ and applying partial fractions we have

$$\begin{aligned}F(s) &= \frac{10s^2+12s+14}{(s^2-2s+2)(s+2)} = \frac{10s^2+12s+14}{((s-1)^2+1)(s+2)} \\ &= \frac{A}{s+2} + \frac{B(s-1)+C}{(s-1)^2+1} = \frac{A}{s+2} + \frac{B(s-1)}{(s-1)^2+1} + \frac{C}{(s-1)^2+1},\end{aligned}$$

so that

$$\begin{aligned}\mathscr{L}^{-1}\{\frac{10s^2+12s+14}{(s^2-2s+2)(s+2)}\} &= A\mathscr{L}^{-1}\{\frac{1}{s+2}\} + B\mathscr{L}^{-1}\{\frac{(s-1)}{(s-1)^2+1}\} + C\mathscr{L}^{-1}\{\frac{1}{(s-1)^2+1}\} \\ &= Ae^{-2t} + Be^{t}\cos t + Ce^{t}\sin t.\end{aligned}$$

(d) Write

$$F(s) = \ln\left(\frac{s^2+9}{s^2+1}\right) = \ln(s^2+9) - \ln(s^2+1)$$

and

$$\begin{aligned}\frac{dF(s)}{ds} &= \frac{d}{ds}\ln(s^2+9) - \frac{d}{ds}\ln(s^2+1)\\ &= \frac{2s}{s^2+9} - \frac{2s}{s^2+1} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+1}\\ &= \frac{As}{s^2+9} + \frac{B}{s^2+9} + \frac{Cs}{s^2+1} + \frac{D}{s^2+1}.\end{aligned}$$

Thus

$$\begin{aligned}\mathscr{L}^{-1}\{\frac{dF(s)}{ds}\} &= A\mathscr{L}^{-1}\{\frac{s}{s^2+9}\} + B\mathscr{L}^{-1}\{\frac{1}{s^2+9}\} + C\mathscr{L}^{-1}\{\frac{s}{s^2+1}\} + D\mathscr{L}^{-1}\{\frac{1}{s^2+1}\}\\ &= A\cos(3t) + \frac{1}{3}B\sin(3t) + C\cos t + D\sin t.\end{aligned}$$

Using $tf(t) = -\mathscr{L}^{-1}\{\frac{dF}{ds}\}$ we find

$$\mathscr{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathscr{L}^{-1}\{\frac{dF(s)}{ds}\} = -\frac{1}{t}(A\cos(3t) + \frac{1}{3}B\sin(3t) + C\cos t + D\sin t).$$

Question 4. Solve the given initial value problem using the method of Laplace transforms. You do not need to determine the constants of the partial fractions.

(a) $y'' - y' - 2y = 0$, $y(0) = -2$, $y'(0) = 5$.

(b) $y'' + y = t$, $y(\pi) = 0$, $y'(\pi) = 0$.

(c) $y'' + 5y' - 6y = 21e^{t-1}$, $y(1) = -1$, $y'(1) = 9$.

Solution 4. (a) Taking the Laplace transform and using its properties we find

$$Y(s) = \frac{-2s+7}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1},$$

so that $y(t) = Ae^{3t} + Be^{-t}$.

(b) We need to shift t so we introduce $w(t) = y(t+\pi)$. Then $w'' + w = t + \pi$ and taking the Laplace transform we find

$$\begin{aligned} W(s) &= \frac{1+\pi s}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} \\ &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs}{s^2+1} + \frac{D}{s^2+1}. \end{aligned}$$

Then

$$w(t) = A + Bt + C\cos t + D\sin t.$$

Thus $y(t) = w(t-\pi) = A + B(t-\pi) + C\cos(t-\pi) + D\sin(t-\pi)$.

(c) This is similar to (b). The answer is $y(t) = Ae^{t-1} + Be^{-6(t-1)}$.

Question 5. Use convolution to obtain a formula for the solution to the given initial value problem, where g is piecewise continuous on $[0, \infty)$ and of exponential order.

(a) $y'' + 9y = g(t)$, $y(0) = 1$, $y'(0) = 0$.

(b) $y'' + 4y' + 5y = g(t)$, $y(0) = 1$, $y'(0) = 1$.

Solution 5. (a) By the superposition principle we can write $y = x+z$, where $x''+9x = g(t)$, $x(0) = 0$, $x'(0) = 0$, and $z'' + 9z = 0$, $z(0) = 1$, $z'(0) = 0$. We fin $z(t) = c_1 \cos(3t) + c_2 \sin(3t)$, where c_1 and c_2 are constants determined from the given initial conditions. Next, we find $h(t)$ as

$$h(t) = \mathscr{L}^{-1}\{\frac{1}{s^2+9}\} = \frac{1}{3}\sin(3t).$$

Then, as done in class, as have

$$y(t) = z(t) + \frac{1}{3}g(t) * \sin(3t).$$

(b) Similar to (a). The answer is $y(t) = z(t) + g(t) * (e^{-2t} \sin t)$.

Question 6. Solve the given integro-differential equation for $y(t)$.

$$
\begin{aligned}
&y'(t) + \int_0^t (t-\tau)y(\tau)\,d\tau = t,\\
&y(0) = 0.
\end{aligned}
$$

Solution 6. Write the equation as

$$y'(t) + t * y(t) = t$$

Then

$$sY(s) + \frac{1}{s^2}Y(s) = \frac{1}{s^2},$$

so that

$$
\begin{aligned}
Y(s) &= \frac{1}{s^3+1} = \frac{1}{(s+1)(s^2-s+1)} = \frac{1}{(s+1)((s-1/2)^2+(\sqrt{3}/2)^2)}\\
&= \frac{1}{3}\frac{1}{s+1} + \frac{1}{3}\frac{2-s}{(s-1/2)^2+(\sqrt{3}/2)^2)}\\
&= \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\frac{s-1/2}{(s-1/2)^2+(\sqrt{3}/2)^2)} + \frac{1}{\sqrt{3}}\frac{\sqrt{3}/2}{(s-1/2)^2+(\sqrt{3}/2)^2)}.
\end{aligned}
$$

Thus

$$y(t) = \frac{1}{3}e^{-t} - \frac{1}{3}e^{\frac{t}{2}}\cos(\frac{\sqrt{3}}{2}t) + \frac{1}{\sqrt{3}}e^{\frac{t}{2}}\sin(\frac{\sqrt{3}}{2}t).$$

Question 7. Solve the given initial value problem.

(a) $y'' + y = \delta(t - \frac{\pi}{2})$, $y(0) = 0$, $y'(0) = 1$.

(b) $y'' + y = \delta(t - \pi) - \delta(t - 2\pi)$, $y(0) = 0$, $y'(0) = 1$.

Solution 7. (a) Applying the Laplace transform we find

$$Y(s) = \frac{1}{s^2 + 1} + e^{-\frac{\pi}{2}s} \frac{1}{s^2 + 1}$$

so that

$$y(t) = \sin t + \sin(t - \frac{\pi}{2}) u(t - \frac{\pi}{2}).$$

(b) Similar to part (a). The answer is $y(t) = \sin t + \sin(t - \pi) u(t - \pi) + \sin(t - 2\pi) u(t - 2\pi)$.

Question 8. Review the homework problems, results proved in class, and examples posted in the course webpage.

Solution 8. N/A.