VANDERBILT UNIVERSITY

MATH 2420 – METHODS OF ORDINARY DIFFERENTIAL EQUATIONS

Practice for test 2

Directions: This practice test should be used as a study guide, illustrating the concepts that will be emphasized in the test. This does not mean that the actual test will be restricted to the content of the practice. Try to identify, from the questions below, the concepts and methods that you should master for the test. For each question in the practice test, study the ideas and techniques connected to the problem, even if they are not directly used in your solution.

Take this also as an opportunity to practice how you will write your solutions in the test. For this, write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

The first test will cover all material discussed from (including) section 6.1 to (including) section 7.9. (Note that sections 1.3 and 1.4 will not be in the test.)

The table below indicates the Laplace transform $F(s)$ of the given function $f(t)$.

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$

The following are the main properties of the Laplace transform.

Function	Laplace transform
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
$e^{at}f(t)$	$F(s-a)$
$t^nf(t)$	$(-1)^n\frac{d^n}{ds^n}F(s)$
$(f*g)(t)$	$F(s)G(s)$
$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)u(t-a)$	$e^{-as}F(s)$

Above, $f*g$ is the convolution of f and g, given by

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau,$$

and $u(t-a)$ is given by

$$u(t-a)=\begin{cases}0, & t<a,\\ 1, & t>a.\end{cases}$$

Question 1. Solve the following differential equations.

(a) $x''' - 3x'' + 4x = 0$.

(b) $x''' - 3x'' - x' + 3x = 0$

(c) $x'''' + 4x'' + 4x = 0$.

Question 2. Recall that a function f is said to be of exponential order $\alpha > 0$ if there exist positive constants T and M such that

$$|f(t)| \leq Me^{\alpha t}, \text{ for all } t \geq T.$$

Which of the following functions are of exponential order?

(a) $t \ln t$.

(b) e^{t^3}.

(c) $\dfrac{1}{t^2+1}$.

Question 3. Determine $\mathscr{L}^{-1}\{F\}$. You do not need to determine the constants of the partial fractions.

(a) $F(s) = \dfrac{5s^2 + 34s + 53}{(s+3)^2(s+1)}$.

(b) $s^2F(s) + sF(s) - 6F(s) = \dfrac{s^2+4}{s^2+s}$.

(c) $sF(s) + 2F(s) = \dfrac{10s^2 + 12s + 14}{s^2 - 2s + 2}$.

(d) $F(s) = \ln\left(\dfrac{s^2+9}{s^2+1}\right)$.

Question 4. Solve the given initial value problem using the method of Laplace transforms. You do not need to determine the constants of the partial fractions.

(a) $y'' - y' - 2y = 0$, $y(0) = -2$, $y'(0) = 5$.

(b) $y'' + y = t$, $y(\pi) = 0$, $y'(\pi) = 0$.

(c) $y'' + 5y' - 6y = 21e^{t-1}$, $y(1) = -1$, $y'(1) = 9$.

Question 5. Use convolution to obtain a formula for the solution to the given initial value problem, where g is piecewise continuous on $[0, \infty)$ and of exponential order.

(a) $y'' + 9y = g(t)$, $y(0) = 1$, $y'(0) = 0$.

(b) $y'' + 4y' + 5y = g(t)$, $y(0) = 1$, $y'(0) = 1$.

Question 6. Solve the given integro-differential equation for $y(t)$.

$$y'(t) + \int_0^t (t - \tau) y(\tau)\, d\tau = t,$$
$$y(0) = 0.$$

Question 7. Solve the given initial value problem.

(a) $y'' + y = \delta(t - \frac{\pi}{2})$, $y(0) = 0$, $y'(0) = 1$.

(b) $y'' + y = \delta(t - \pi) - \delta(t - 2\pi)$, $y(0) = 0$, $y'(0) = 1$.

Question 8. Review the homework problems, results proved in class, and examples posted in the course webpage.