VANDERBILT UNIVERSITY

MATH 2420 – METHODS OF ORDINARY DIFFERENTIAL EQUATIONS

Practice final solutions

Directions: This practice final should be used as a study guide, illustrating the concepts that will be emphasized in the final exam. This does not mean that the actual test will be restricted to the content of the practice. Try to identify, from the questions below, the concepts and methods that you should master for the final exam. For each question in the practice final, study the ideas and techniques connected to the problem, even if they are not directly used in your solution.

Take this also as an opportunity to practice how you will write your solutions in the test. For this, write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

The final exam will cover all material, with emphasis on chapter 8, with the following exceptions. Sections 1.3, 1.4, the Cauchy-Euler equation, the Laplace transform of periodic functions, and the Gamma function will not be in the test. Section 8.8 will be on the test only as an extra credit question.

If you need to perform a partial fraction decomposition to solve a problem, you do not need to find the constants of the partial fractions.

The Laplace transforms and series given below are going to be given in the test.

The table below indicates the Laplace transform $F(s)$ of the given function $f(t)$:

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$

The following are the main properties of the Laplace transform:

Function	Laplace transform
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
$e^{at}f(t)$	$F(s-a)$
$t^nf(t)$	$(-1)^n\frac{d^n}{ds^n}F(s)$
$(f*g)(t)$	$F(s)G(s)$
$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)u(t-a)$	$e^{-as}F(s)$

Above, $f*g$ is the convolution of f and g, given by

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau,$$

and $u(t-a)$ is given by

$$u(t-a)=\begin{cases}0, & t<a,\\ 1, & t>a.\end{cases}$$

Below are some common Taylor series:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!},$$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n},$$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1},$$

$$\ln x=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n.$$

Question 1. For each equation below, identify the unknown function, classify the equation as linear or non-linear, and state its order.

(a) $x''' - x^2 = 0$.

(b) $y'' - x^3 y' = e^x$.

(c) $u' + u = \cos u$.

Solution 1. (a) unknown: x, non-linear, third order. (b) unknown: y, linear, second order. (c) unknown: u, non-linear, first order.

Question 2. For each equation below, list all methods learned in class that can be used to find a solution.

(a) $x''' - x' + 2x = 0$.

(b) $x' + tx = e^t$.

(c) $x' + \sin t x = 0$.

(d) $t^3 \cos x x' = -3t^2 \sin x$

(e) $x'' + x = \delta(t)$.

(f) $t^2 x'' - te^t x' + \sin t x = 0$.

Solution 2. (a) Characteristic equation, Laplace transform, power series. (b) Formula for first order linear equations. (c) Separable equation, formula for first order linear equations, power series. (d) Separable equation, exact equation. (e) Laplace transform. (f) Power series.

Question 3. Solve the equations below.

(a) $y' - y^2 = 0$.

(b) $x' + \sin t x = 0$.

(c) $(1 + t^2)x'' + 3tx' - x = 0$.

(d) $x'' + x = f(t)$, $x(0) = 2$, $x'(0) = 0$,

where

$$f(t) = \begin{cases} -t, & 0 < t < 2, \\ 3, & t > 2. \end{cases}$$

(e) $x''' - 2x'' + x' - 2x = 0$.

Solution 3. (a) This equation is separable. Write $\frac{dy}{y^2} = dt$, $y \neq 0$, which gives $y = \frac{1}{C-x}$; $y = 0$ is also a solution.

(b) This equation is separable, we find $x = Ce^{\cos t}$.

(c) $t = 0$ is an ordinary point of this equation, so we look for a power series solution about zero. Plugging in $x(t) = \sum_{n=0}^{\infty} a_n t^n$ we find

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n t^n + 3\sum_{n=0}^{\infty} na_n t^n - \sum_{n=0}^{\infty} a_n t^n = 0.$$

Noting that we can start the first sum at $n = 2$ and shifting its summation index we find

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n(n-1) + 3n - 1)a_n\right] t^n = 0,$$

which gives the recurrence relation

$$a_{n+2} = -\frac{n(n-1) + 3n - 1}{(n+2)(n+1)} a_n,\ n \geq 0.$$

The solution is given by $x(t) = \sum_{n=0}^{\infty} a_n t^n$, with a_0 and a_1 arbitrary, and the remaining a_n's determined by the above recurrence relation.

(d) Write $f(t) = -t\Pi_{0,2}(t) + 3u(t-2) = -tu(t) + (t+3)u(t-2) = -tu(t) + (t+5-2)u(t-2)$. Taking the Laplace transform we find

$$s^2 X(s) - 2s + X(s) = -\frac{1}{s^2} + e^{-2s}(\frac{1}{s^2} + \frac{5}{s}).$$

Solving for $X(s)$ and using partial fractions

$$\begin{aligned} X(s) &= \frac{2s}{s^2+1} - \frac{1}{s^2(s^2+1)} + e^{-2s}(\frac{1}{s^2(s^2+1)} + \frac{5}{s(s^2+1)}) \\ &= \frac{As+B}{s^2+1} + \frac{C}{s} + \frac{D}{s^2} + e^{-2s}(\frac{E}{s} + \frac{F}{s^2} + \frac{Gs+H}{s^2+1}), \end{aligned}$$

from which

$$x(t) = A\cos t + B\sin t + C + Dt + u(t-2)(E + F(t-2) + G\cos(t-2) + H\sin(t-2)).$$

(e) The characteristic equation is $\lambda^3 - 2\lambda^2 + \lambda - 2 = (\lambda^2+1)(\lambda-2) = 0$ so that $\lambda = \pm i$ and $\lambda = 2$ are the roots. Thus $x(t) = c_1 \cos t + c_2 \sin t + c_3 e^{2t}$.

Question 4. Explain why the equation

$$(1+t^2)x'' + \sin t x' + \cos t x = 0$$

admits a power series solution centered at zero. Find the first four non-zero terms of this power series solution. What can you say about its radius of convergence?

Solution 4. The functions $\frac{\sin t}{t^2+1}$ and $\frac{\cos t}{t^2+1}$ are both analytic at zero, thus zero is an ordinary point. Setting $x(t) = \sum_{n=0}^{\infty} a_n t^n$, plugging it in, and using the series for sine and cosine gives

$$\begin{aligned} \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n t^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1} \sum_{n=0}^{\infty} na_n t^{n-1} \\ + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} \sum_{n=0}^{\infty} a_n t^n = 0. \end{aligned}$$

Starting the first sum at $n=2$, shifting its summation index, combining the resulting expression with the second sum, and expanding the sums in the last two terms gives

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+n(n-1)a_n\right]t^n+\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\dots\right)(a_1+2a_2t+3a_3t^2+\dots)$$
$$+\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\dots\right)(a_0+a_1t+a_2t^2+\dots)=0.$$

Grouping the terms with the same power

$$t^0(2\cdot 1a_2+a_0)+t^1(3\cdot 2a_3+a_1+a_1)+t^2(4\cdot 3a_4+2\cdot 1a_2+2a_2+a_2-\frac{1}{2!}a_0)+\dots=0.$$

This gives $a_2=-\frac{1}{2}a_0$, $a_3=-\frac{1}{3}a_1$, $a_4=-\frac{5}{12}a_2+\frac{1}{24}a_0=\frac{1}{4}a_0$. Thus

$$\begin{aligned}x(t)&=a_0+a_1t+a_2t^2+a_3t^3+a_4t^4+\dots\\&=a_0+a_1t-\frac{1}{2}a_0t^2+-\frac{1}{3}a_1t^3+\frac{1}{24}a_0t^4+\dots\\&=a_0(1-\frac{1}{2}t^2+\frac{1}{24}t^4+\dots)+a_1(t-\frac{1}{3}a_1t^3+\dots),\end{aligned}$$

where a_0 and a_1 are arbitrary. The singular points are $\pm i$, implying that the radius of convergence is at least 1.

Question 5. Find all the singular points of

$$(1+t)t^2x''+(1-t^2)x'-x=0.$$

Write the form of a power series solution centered at -1 and use it to find a recurrence relation for its coefficients. Write the form of a second, linearly independent, solution.

Solution 5. The singular points are $t=0$ and $t=-1$. For $x_0=-1$, we readily find $p_0=q_0=0$ and the indicial equation is $r(r-1)=0$, with roots $r_1=1$, $r_2=0$. Rewrite the equation as

$$\begin{aligned}(1+t)t^2x''+(1-t^2)x'-x&=(1+t)(t+1-1)^2x''+(1+t)(1-t)x'-x\\&=(1+t)[(t+1)^2-2(t+1)+1]x''+(1+t)[2-(t+1)]x'-x\\&=[(t+1)^3-2(t+1)^2+(t+1)]x''+[-(t+1)^2+2(t+1)]x'-x=0.\end{aligned}$$

Set $x(t)=\sum_{n=0}^{\infty}a_n(t+1)^{n+r_1}=\sum_{n=0}^{\infty}a_n(t+1)^{n+1}$ and plug in to find

$$\sum_{n=0}^{\infty}(n+1)na_n(t+1)^{n+2}+\sum_{n=0}^{\infty}(-2)(n+1)na_n(t+1)^{n+1}+\sum_{n=0}^{\infty}(n+1)na_n(t+1)^n$$
$$+\sum_{n=0}^{\infty}(-1)(n+1)a_n(t+1)^{n+2}+\sum_{n=0}^{\infty}2(n+1)(t+1)^{n+1}+\sum_{n=0}^{\infty}(-1)a_n(t+1)^{n+1}=0.$$

Grouping the first and fourth sums and the second, fifth, and last sums, we find

$$\sum_{n=0}^{\infty}(n+1)(n-1)a_n(t+1)^{n+2}+\sum_{n=0}^{\infty}[2(n+1)(1-n)-1]a_n(t+1)^{n+1}+\sum_{n=0}^{\infty}(n+1)na_n(t+1)^n=0.$$

The $n=0$ term in the last sum vanishes (as it must, it corresponds to the $(r(r-1)+p_0r+q_0)a_0$ term). Expanding the second sum up to $n=1$ and shifting the summation index in the resulting expression, expanding the last sum up to $n = 2$, and shifting the summation index in the first sum, we find

$$(a_0 + 2a_1)(t+1) + \sum_{n=2}^{\infty}(n-1)(n-3)a_{n-2}(t+1)^n + \sum_{n=2}^{\infty}[2n(2-n)-1]a_{n-1}t^n + \sum_{n=2}^{\infty}(n+1)na_n t^n = 0.$$

We find $a_1 = -\frac{1}{2}a_0$ and

$$a_n = \frac{(n-1)(n-3)a_{n-2} + 2n(2-n) - 1]a_{n-1}}{n(n+1)}, \; n \geq 2.$$

Because $r_1 - r_2 = 1$, the second linearly independent solution has the form

$$x_2(t) = A\ln(t+1)x_1(t) + \sum_{n=0}^{\infty} b_n(t+1)^{n+0}, \; b_0 \neq 0.$$

Question 6. Review the first two tests and practice tests. Be prepared to state definitions and the statement of theorems discussed in class.

Solution 6. N/A.