VANDERBILT UNIVERSITY

MATH 2420 – METHODS OF ORDINARY DIFFERENTIAL EQUATIONS

Practice final

Directions: This practice final should be used as a study guide, illustrating the concepts that will be emphasized in the final exam. This does not mean that the actual test will be restricted to the content of the practice. Try to identify, from the questions below, the concepts and methods that you should master for the final exam. For each question in the practice final, study the ideas and techniques connected to the problem, even if they are not directly used in your solution.

Take this also as an opportunity to practice how you will write your solutions in the test. For this, write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

The final exam will cover all material, with emphasis on chapter 8, with the following exceptions. Sections 1.3, 1.4, the Cauchy-Euler equation, the Laplace transform of periodic functions, and the Gamma function will not be in the test. Section 8.8 will be on the test only as an extra credit question.

If you need to perform a partial fraction decomposition to solve a problem, you do not need to find the constants of the partial fractions.

The Laplace transforms and series given below are going to be given in the test.

The table below indicates the Laplace transform $F(s)$ of the given function $f(t)$:

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$

The following are the main properties of the Laplace transform:

Function	Laplace transform
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
$e^{at}f(t)$	$F(s-a)$
$t^nf(t)$	$(-1)^n\frac{d^n}{ds^n}F(s)$
$(f*g)(t)$	$F(s)G(s)$
$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)u(t-a)$	$e^{-as}F(s)$

Above, $f*g$ is the convolution of f and g, given by

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau,$$

and $u(t-a)$ is given by

$$u(t-a)=\begin{cases}0, & t<a,\\ 1, & t>a.\end{cases}$$

Below are some common Taylor series:

$$e^x=\sum_{n=0}^{\infty}\frac{x}{n!},$$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n},$$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1},$$

$$\ln x=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n.$$

Question 1. For each equation below, identify the unknown function, classify the equation as linear or non-linear, and state its order.

(a) $x''' - x^2 = 0$.

(b) $y'' - x^3 y' = e^x$.

(c) $u' + u = \cos u$.

Question 2. For each equation below, list all methods learned in class that can be used to find a solution.

(a) $x''' - x' + 2x = 0$.

(b) $x' + tx = e^t$.

(c) $x' + \sin tx = 0$.

(d) $t^3 \cos x x' = -3t^2 \sin x$

(e) $x'' + x = \delta(t)$.

(f) $t^2 x'' - te^t x' + \sin tx = 0$.

Question 3. Solve the equations below.

(a) $y' - y^2 = 0$.

(b) $x' + \sin tx = 0$.

(c) $(1 + t^2)x'' + 3tx' - x = 0$.

(d) $x'' + x = f(t)$, $x(0) = 2$, $x'(0) = 0$,
where

$$f(t) = \begin{cases} -t, & 0 < t < 2, \\ 3, & t > 2. \end{cases}$$

(e) $x''' - 2x'' + x' - 2x = 0$.

Question 4. Explain why the equation

$$(1 + t^2)x'' + \sin tx' + \cos tx = 0$$

admits a power series solution centered at zero. Find the first four non-zero terms of this power series solution. What can you say about its radius of convergence?

Question 5. Find all the singular points of

$$(1 + t)t^2 x'' + (1 - t^2)x' - x = 0.$$

Write the form of a power series solution centered at -1 and use it to find a recurrence relation for its coefficients. Write the form of a second, linearly independent, solution.

Question 6. Review the first two tests and practice tests. Be prepared to state definitions and the statement of theorems discussed in class.