

## VANDERBILT UNIVERSITY

### MATH 2420 – METHODS OF ORDINARY DIFFERENTIAL EQUATIONS

#### *Examples of section 2.2*

**Question 1.** The intensity  $I$  of the light at a depth of  $x$  meters below the surface of a lake satisfies the differential equations  $I' = -1.4I$ .

- (a) At what depth is the intensity half of the intensity  $I_0$  at the surface of the water?
- (b) At what depth will the intensity be 1 % of that at the surface?

**Question 2.** According to one cosmological theory, there were equal amounts of the two uranium isotopes  $^{235}\text{U}$  and  $^{238}\text{U}$  at the creation of the universe in the big bang. At present there are 137.7 atoms of  $^{238}\text{U}$  for each atom of  $^{235}\text{U}$ . Using the half-lives  $4.51 \times 10^9$  years for  $^{238}\text{U}$  and  $7.10 \times 10^8$  years for  $^{235}\text{U}$ , and assuming a radioactive decay model of the form  $x' = kx$ , calculate the age of the universe.

#### SOLUTIONS.

**1a.** The differential equation is of the form  $x' = kx$ . This equation is separable, and can be written as

$$\frac{dx}{x} = k dt,$$

provided that  $x \neq 0$ . Integrating

$$\int \frac{dx}{x} = k \int dt \Rightarrow \ln |x| = kt + C,$$

where  $C$  is an arbitrary constant of integration. Thus,

$$|x| = e^C e^{kt}.$$

We can remove the absolute value by introducing a sign, i.e., the above tells us that

$$x = \pm e^C e^{kt}.$$

Since  $C$  is an arbitrary constant, so is  $e^C$ , and therefore we can set  $A = \pm e^C$  for some constant  $A$ . Our solution then reads

$$x(t) = A e^{kt}.$$

Next, we notice that the constant  $A$  has a very simple interpretation: if  $x_0$  is the value of  $x$  at  $t = 0$ , i.e.,  $x(0) = x_0$ , then

$$x(0) = x_0 = A e^{k0} \Rightarrow A = x_0,$$

and therefore we can write the solution as

$$x(t) = x_0 e^{kt}.$$

Now we turn our attention to the problem in question. We have that the intensity at a depth of  $x$  meters is  $I(x) = I_0 e^{-1.4x}$ . Then

$$I(x) = \frac{I_0}{2} = I_0 e^{-1.4x} \Rightarrow x = \frac{\ln 2}{1.4} \approx 0.495 \text{ meters.}$$

**1b.** Solving  $I_0 e^{-1.4x} = 0.01 I_0$  for  $x$  gives  $x = \frac{\ln 100}{1.4} \approx 3.29$  meters.

**2.** Let  $N_8(t)$  and  $N_5(t)$  be the numbers of  $^{238}\text{U}$  and  $^{235}\text{U}$  atoms, respectively,  $t$  billions of years after the big bang. Since both isotopes follow a radioactive decay model  $x' = kx$ , whose solution is (see previous problem)  $x(t) = x_0 e^{kt}$ , we have

$$N_8 = N_0 e^{-kt},$$

and

$$N_5 = N_0 e^{-\ell t},$$

where  $N_0$  is the initial number of atoms of each isotope, which is the same for both  $^{238}\text{U}$  and  $^{235}\text{U}$  by hypothesis. Notice however that the rates of decay,  $k$  and  $\ell$ , differ for these isotopes. Their values are given by

$$\begin{aligned} N_8(4.51) &= \frac{N_0}{2} = N_0 e^{-k \times 4.51} \Rightarrow k = \frac{\ln 2}{4.51}, \\ N_5(0.71) &= \frac{N_0}{2} = N_0 e^{-\ell \times 0.71} \Rightarrow \ell = \frac{\ln 2}{0.71}. \end{aligned}$$

We know that for the value of  $t$  corresponding to “now” we have  $\frac{N_8}{N_5} = 137.7$ , hence

$$\frac{N_8}{N_5} = \frac{N_0 e^{-kt}}{N_0 e^{-\ell t}} = e^{(\ell-k)t} = e^{(\frac{\ln 2}{0.71} - \frac{\ln 2}{4.51})t} = 137.7.$$

Solving for  $t$  gives

$$t = \frac{\ln 137.7}{\frac{\ln 2}{0.71} - \frac{\ln 2}{4.51}} \approx 5.99.$$

According to this theory, therefore, the universe should be about 6 billion years old.

**Note:** According to our best current models, the age of the universe is estimated to be about 13.7 billions of years, and the initial ratio of  $^{235}\text{U}$  to  $^{238}\text{U}$  is estimated to be 1.65 rather than one, as in the exercise<sup>1</sup>. See S. Weinberg, *Cosmology*, Oxford University Press. The interested student can consult the non-technical book *The First Three Minutes: A Modern View Of The Origin Of The Universe*, by the same author.

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<sup>1</sup>It makes sense that it is larger than one because three additional neutrons must be added to the progenitor of  $^{235}\text{U}$  to form the progenitor of  $^{238}\text{U}$ .