VANDERBILT UNIVERSITY
MATH 234 — INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS
PRACTICE FINAL SOLUTIONS.

Disclaimer: I cannot guarantee that these solutions are typo-free!

Question 1. Solve the following initial-boundary value problem

$$\begin{cases} u_{sr} = 0, & s \geq -r, -\infty < r < \infty, \\ u(-r, r) = F(r), & -\infty < r < \infty, \\ u_s(-r, r) = G(r), & -\infty < r < \infty, \end{cases}$$

where $u = u(s, r)$ is the unknown, and F and G are given C^∞ functions.
Hint: Change variables $t = s + r$, $x = s - r$.
Solution. Let

$$\begin{cases} t = s + r, & \text{(1a)} \\ x = s - r, & \text{(1b)} \end{cases}$$

which gives

$$\begin{cases} t_s = 1, & \text{(2a)} \\ t_r = 1, & \text{(2b)} \\ x_s = 1, & \text{(2c)} \\ x_r = -1, & \text{(2d)} \end{cases}$$

where t_s is $\frac{\partial t}{\partial s}$, and so on. Let

$$u(s, r) = u(t(s, r), x(s, r)).$$

Using the chain rule,

$$\begin{aligned} u_s &= u_t t_s + u_x x_s \\ &= u_t + u_x, \end{aligned}$$

where we used (2a) and (2c). Differentiating again and using (2b) and (2d),

$$\begin{aligned} u_{sr} &= u_{tt} t_r + u_{tx} x_r + u_{xt} t_r + u_{xx} x_r \\ &= u_{tt} - u_{tx} + u_{xt} - u_{xx} \\ &= u_{tt} - u_{xx}. \end{aligned}$$

Thus, $u_{sr} = 0 \Leftrightarrow u_{tt} - u_{xx} = 0$, and we see that the original equation is simply the wave equation written in a different set of coordinates. For the initial conditions, we see from (1) that

$$u(-r, r) = u(0, -\frac{x}{2}),$$

and

$$u_s(-r, r) = u_t(0, -\frac{x}{2}) + u_x(0, -\frac{x}{2}).$$

Hence,

$$u(0, x) = F(x),\ u_t(0, x) = G(x) - F_x(x).$$

From these formulas the solution is now easily found via an application of D'Alembert's formula.

Question 2. Let $u(t,x)$ be a solution to the following initial-value problem:

$$\begin{cases} u_{tt} - u_{xx} = f(t,x), & -\infty < x < \infty,\, t > 0, \\ u(0,x) = g(x),\, u_t(0,x) = h(x), & -\infty < x < \infty, \end{cases}$$

where f, g, and h are C^∞ functions. Assume that there exist numbers X, Y, and Z, such that

$$|f(t,x)| \le X,\, |g(x)| \le Y,\, |h(x)| \le Z,$$

for all $t \ge 0$, $x \in \mathbb{R}$. Show that for any $t > 0$, and any $x \in \mathbb{R}$, it holds that

$$|u(t,x)| \le Y + tZ + \frac{1}{2}Xt^2.$$

Hint: D'Alembert and Duhamel.

Solution. By uniqueness, u can be written as

$$u = w + v, \tag{3}$$

where w and v are, respectively, solutions to

$$\begin{cases} v_{tt} - v_{xx} = 0, & -\infty < x < \infty,\, t > 0, \\ v(0,x) = g(x),\, v_t(0,x) = h(x), & -\infty < x < \infty, \end{cases} \tag{4}$$

and

$$\begin{cases} w_{tt} - w_{xx} = f(t,x), & -\infty < x < \infty,\, t > 0, \\ w(0,x) = 0,\, w_t(0,x) = 0, & -\infty < x < \infty. \end{cases} \tag{5}$$

Problem (4) is solved with D'Alembert's formula

$$v(t,x) = \frac{g(x-t) + g(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy,$$

so that

$$\begin{aligned} |v(t,x)| &\le \frac{1}{2}|g(x-t)| + \frac{1}{2}|g(x+t)| + \frac{1}{2}\int_{x-t}^{x+t} |h(y)|\,dy \\ &\le \frac{1}{2}Y + \frac{1}{2}Y + \frac{1}{2}\int_{x-t}^{x+t} Z\,dy \\ &= Y + tZ. \end{aligned} \tag{6}$$

Problem (5) is solved using Duhamel's formula, i.e., w is given by

$$w(t,x) = \int_0^t z(t-s,s,x)\,ds, \tag{7}$$

where $z = z(t,s,x)$ solves

$$\begin{cases} z_{tt} - z_{xx} = 0, & -\infty < x < \infty,\, t > 0, \\ z(0,s,x) = 0,\, z_t(0,s,x) = f(s,x), & -\infty < x < \infty. \end{cases} \tag{8}$$

Problem (8), in turn, is solved via an application of D'Alembert's formula,

$$z(t,s,x) = \frac{1}{2}\int_{x-t}^{x+t} f(s,y)\,dy.$$

Then

$$\begin{aligned} |z(t,s,x)| &\leq \frac{1}{2}\int_{x-t}^{x+t} |f(s,y)|\,dy \\ &\leq Xt \end{aligned} \tag{9}$$

Combining (7) and (9), we find

$$\begin{aligned} |w(t,x)| &\leq \int_0^t |z(t-s,s,x)|\,ds \\ &\leq X\int_0^t (t-s)\,ds \\ &= \frac{1}{2}Xt^2. \end{aligned} \tag{10}$$

Combining (3), (6), and (10) yields the result.

Question 3. Let v and w be, respectively, solutions to

$$\begin{cases} v_{tt} - \Delta v = 0, & x \in \mathbb{R}^n,\ t > 0, \\ v(0,x) = f_1(x),\ v_t(0,x) = g_1(x), & x \in \mathbb{R}^n, \end{cases}$$

and

$$\begin{cases} w_{tt} - \Delta w = 0, & x \in \mathbb{R}^n,\ t > 0, \\ w(0,x) = f_2(x),\ w_t(0,x) = g_2(x), & x \in \mathbb{R}^n, \end{cases}$$

where f_1, f_2, g_1, and g_2 are given smooth functions. Suppose that $f_1(x) = f_2(x)$ for all $x \in B_1(0)$, and $g_1(x) = g_2(x)$ for all $x \in B_1(0)$. Show that $v(t,x) = w(t,x)$ for all $(t,x) \in C$, where C is the cone

$$C = \Big\{ (t,x) \in [0,\infty) \times \mathbb{R}^n \,\Big|\, 0 \le t \le 1,\ |x| \le 1 - t \Big\}.$$

Hint: Use

$$E(t) = \frac{1}{2} \int_{B_{1-t}(0)} \Big[(\partial_t u(t,x))^2 + c^2 |\nabla u(t,x)|^2 \Big] \, dx,$$

where ∇ is the gradient in $\mathbb{R}^n$ and $|\nabla u(t,x)|$ is the norm of the vector $\nabla u(t,x)$.

Solution. It suffices to show that if $u(0,x) = 0$ and $u_t(0,x) = 0$, for $|x| \le 1$, then $u = 0$ on C. Define for $0 \le t \le 1$,

$$E(t) = \frac{1}{2} \int_{B_{1-t}(0)} \big(u_t^2(t,x) + |\nabla u(t,x)|^2 \big) dx.$$

Differentiating and integrating by parts,

$$\begin{aligned} \frac{d}{dt} E(t) &= \int_{B_{1-t}(0)} \big(u_t u_{tt} + \langle \nabla u, \nabla u_t \rangle \big) dx - \frac{1}{2} \int_{\partial B_{1-t}(0)} \big(u_t^2 + |\nabla u|^2 \big) ds \\ &= \int_{B_{1-t}(x_0)} u_t (u_{tt} - \Delta u) + \int_{\partial B_{1-t}(0)} \frac{\partial u}{\partial \nu} u_t \, ds - \frac{1}{2} \int_{\partial B_{1-t}(0)} \big(u_t^2 + |\nabla u|^2 \big) ds \\ &= \int_{\partial B_{1-t}(0)} \big(\frac{\partial u}{\partial \nu} u_t - \frac{1}{2} u_t^2 - \frac{1}{2} |\nabla u|^2 \big) ds, \end{aligned}$$

where in the last step we used $u_{tt} - \Delta u = 0$. Since $ab \le \frac{a^2}{2} + \frac{b^2}{2}$ (because $(a-b)^2 \ge 0$),

$$\left| \frac{\partial u}{\partial \nu} u_t \right| \le |u_t| |\nabla u| \le \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2.$$

We conclude that $\frac{d}{dt} E(t) \le 0$, hence $E(t) \le E(0) = 0$ for all $0 \le t \le 1$. It follows that u_t and ∇u vanish identically and so does u within C.

Question 5. Prove the following improved version of the maximum principle. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. If $u \in C^2(\overline{\Omega})$ satisfies

$$\Delta u \geq 0 \text{ in } \Omega,$$

then u attains its maximum on the boundary, i.e.,

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Hint: Assume first that $\Delta u > 0$, and show that this cannot happen if u has a local maximum in Ω. For the case $\Delta u \geq 0$, set $u_\varepsilon = u + \varepsilon e^{x_1}$, where $\varepsilon > 0$, and conclude that $\Delta u_\varepsilon > 0$. Obtain the result by taking the limit $\varepsilon \to 0^+$.

Formulate, and prove, a similar statement for the minimum of u.

Solution. Suppose first that u satisfies $\Delta u > 0$. Let $x_0 \in \overline{\Omega}$ be a point where u attains its maximum (which exists since $\overline{\Omega}$ is compact). If x_0 were an interior point, i.e., $x_0 \in \Omega$, then x_0 would in particular be an interior local maximum, and would satisfy

$$\Delta u(x_0) \leq 0,$$

which is contrary to $\Delta u > 0$. Thus x_0 must be on the boundary.

Consider now the original case, $\Delta u \geq 0$. Then

$$\begin{aligned}
\Delta u_\varepsilon &= \Delta u + \varepsilon \Delta e^{x_1} \\
&= \Delta u + \varepsilon e^{x_1} \\
&> \Delta u \\
&> 0,
\end{aligned}$$

where $u_\varepsilon = u + \varepsilon e^{x_1}$, $\varepsilon > 0$, and we used that $\varepsilon e^{x_1} > 0$ and the assumption $\Delta u \geq 0$. Thus $\Delta u_\varepsilon > 0$, and by the above u_ε cannot attain its maximum in the interior Ω. If u is constant then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

obviously holds, so let us assume that u is not constant. In this case, we claim that that u cannot attain its maximum at an interior point. Indeed, if u had a maximum at $x_0 \in \Omega$, then, by choosing ε sufficiently small, u_ε would also have a maximum at an interior point, which is ruled out by the above. Thus the maximum of u must be on the boundary.

Changing u by $-u$, we obtain that if $\Delta u \leq 0$, then u attains its minimum on the boundary.

Question 6. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the Dirichlet problem for the Laplacian

$$\begin{cases} \Delta u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where $f : \overline{\Omega} \to \mathbb{R}$, and $g : \partial\Omega \to \mathbb{R}$ are smooth (i.e., C^∞) functions. Show that this problem is well-posed.
Hint: For existence, you can simply quote the results from class. For continuous dependence on the parameters, use uniqueness to show that one can write $u = v + w$, where v solves (11) with $g = 0$, and w solves (11) with $f = 0$. Next, define the functions

$$v_+ = (e^{2\alpha d} - e^{\alpha(x_1+d)}) \max_{\overline{\Omega}} |f|,$$

and

$$v_- = -(e^{2\beta d} - e^{\beta(x_1+d)}) \max_{\overline{\Omega}} |f|.$$

Show that for suitable choice of the constants α, β, and d, one can apply the result of question 5 to the functions $v_+ - v$ and $v_- - v$ to conclude that

$$|v| \leq C \max_{\overline{\Omega}} |f|, \tag{12}$$

for some constant C depending on α, β, and d. Finally, use (12) to conclude that u depends continuously on the data of the problem.
Solution. In light of the results developed in class, it suffices to show (12). Let v_- and v_+ be as above, where d is any constant satisfying $|x_1| < d$ for all $x \in \Omega$, and $\alpha > 0$ and $\beta > 0$ are constants that will be determined below. Notice that with this choice of d, $v_+ \geq 0$ and $v_- \leq 0$, for any $\alpha, \beta > 0$. Compute

$$\begin{aligned} \Delta(v_+ - v) &= \Delta\Big((e^{\alpha d} - e^{\alpha(x_1+d)}) \max_{\overline{\Omega}} |f|\Big) - \Delta v \\ &= -\max_{\overline{\Omega}} |f| \Delta e^{\alpha(x_1+d)} - f \\ &= -\alpha^2 e^{\alpha(x_1+d)} \max_{\overline{\Omega}} |f| - f. \end{aligned} \tag{13}$$

Since $|x_1| < d$ and α will be chosen positive, we have that

$$e^{\alpha(x_1+d)} \geq 1.$$

Therefore, if α is chosen sufficiently large, $-\alpha^2 e^{\alpha(x_1+d)} \leq -1$, and (13) implies

$$\Delta(v_+ - v) \leq 0.$$

Therefore, by the previous question, $v_+ - v$ attains its minimum on the boundary, and we conclude

$$\begin{aligned} v_+ - v &\geq \min_{\overline{\Omega}} (v_+ - v) \\ &= \min_{\partial\Omega} (v_+ - v) \\ &= \min_{\partial\Omega} v_+ \\ &\geq 0, \end{aligned}$$

where in the next-to-the-last step we used that v vanishes on the boundary, and in the last step we used that v_+ is non-negative. Therefore $v \le v_+$, or, explicitly,

$$\begin{aligned} v &\le (e^{2\alpha d} - e^{\alpha(x_1+d)}) \max_{\overline{\Omega}} |f| \\ &\le e^{2\alpha d} \max_{\overline{\Omega}} |f|. \end{aligned}$$

This shows that

$$v \le C_1 \max_{\overline{\Omega}} |f|, \tag{14}$$

where $C_1 = e^{2\alpha d}$.

Next, compute

$$\begin{aligned} \Delta(v_- - v) &= \Delta\Big(- (e^{\beta d} - e^{\beta(x_1+d)}) \max_{\overline{\Omega}} |f| \Big) - \Delta v \\ &= \max_{\overline{\Omega}} |f| \Delta e^{\beta(x_1+d)} - f \\ &= \beta^2 e^{\beta(x_1+d)} \max_{\overline{\Omega}} |f| - f. \end{aligned} \tag{15}$$

Since $|x_1| < d$ and β will be chosen positive, we have that

$$e^{\beta(x_1+d)} \ge 1.$$

Therefore, if β is chosen sufficiently large, $\beta^2 e^{\beta(x_1+d)} \ge 1$, and (15) implies that

$$\Delta(v_- - v) \ge 0.$$

Invoking again the previous question, we have that $v_- - v$ attains its maximum on the boundary, and so

$$\begin{aligned} v_- - v &\le \max_{\overline{\Omega}} (v_- - v) \\ &= \max \partial\Omega (v_- - v) \\ &= \max_{\partial\Omega} v_- \\ &\le 0, \end{aligned}$$

where in the next-to-the-last step we used that v vanishes on the boundary, and in the last step we used that v_- is non-positive. Therefore, $v \ge v_-$, or, explicitly,

$$\begin{aligned} v &\ge -(e^{2\beta d} - e^{\beta(x_1+d)}) \max_{\overline{\Omega}} |f| \\ &\ge -e^{2\beta d} \max_{\overline{\Omega}} |f|. \end{aligned} \tag{16}$$

This shows that

$$v \ge -C_2 \max_{\overline{\Omega}} |f|, \tag{17}$$

where $C_2 = e^{2\beta d}$. Setting $C = \max\{C_1, C_2\}$, (14) and (17) give

$$-C \max_{\overline{\Omega}} |f| \le v \le C \max_{\overline{\Omega}} |f|,$$

which means

$$|v| \le C \max_{\overline{\Omega}} |f|,$$

as desired.

Question 7. Prove the result of the previous question using the Green function.
Solution. As in question 6, the results discussed in class allow us to consider only the case when u solves

$$\begin{cases} \Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and it is enough to establish the inequality

$$|u| \leq C \max_{\overline{\Omega}} |f|,$$

for some constant C. Using the representation formula

$$u(x) = -\int_{\Omega} G(x,y) f(y)\, dy,$$

for any $x \in \Omega$. A boundary integral does not appear in this expression because u vanishes on the boundary. Notice also that there is a sign difference from the formula derived in class, since in that case we studied $-\Delta u = f$.

From the above, it follows that

$$\begin{aligned} |u(x)| &= \left| \int_{\Omega} G(x,y) f(y)\, dy \right| \\ &\leq \int_{\Omega} |G(x,y)|\, |f(y)|\, dy \\ &\leq \max_{\overline{\Omega}} |f| \int_{\Omega} |G(x,y)|\, dy. \end{aligned}$$

Recall that $G(x,y) = \Gamma(x-y) + h(y)$, where $\Gamma(x-y)$ is the fundamental solution for the Laplacian and h is a harmonic function in Ω that equals $-\Gamma$ on $\partial\Omega$. This gives

$$|u(x)| \leq M \max_{\overline{\Omega}} |f| \int_{\Omega} \frac{1}{|y-x|^{n-2}}\, dy + M \max_{\overline{\Omega}} |f| \operatorname{vol}(\Omega), \tag{18}$$

for some constant M, and where vol means volume.

Next, choose some $R > 0$ such that $\Omega \subset B_R(x)$, and estimate as follows:

$$\begin{aligned} \int_{\Omega} \frac{1}{|y-x|^{n-2}}\, dy &\leq \int_{B_R(x)} \frac{1}{|y-x|^{n-2}}\, dy \\ &= \int_{B_R(0)} \frac{1}{|y|^{n-2}}\, dy \\ &= \int_{S^{n-1}} \left(\int_0^R \frac{1}{r^{n-2}} r^{n-1}\, dr \right) d\omega \\ &= \frac{R^2}{2} \int_{S^{n-1}} d\omega \\ &= \frac{\operatorname{vol}(S^{n-1}) R^2}{2}. \end{aligned}$$

Setting

$$C = M \left(\operatorname{vol}(\Omega) + \frac{\operatorname{vol}(S^{n-1}) R^2}{2} \right),$$

(18) produces

$$|u| \leq C \max_{\overline{\Omega}} |f|,$$

as desired.

Question 8. Let $\mathbb{R}^2_+$ be the upper half plane in $\mathbb{R}^2$, i.e.,

$$\mathbb{R}^2_+ = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, y > 0 \right\}.$$

Consider the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ u = 0 & \text{for } y = 0 \\ \frac{\partial u}{\partial y} = \frac{1}{n}\sin(nx) & \text{for } y = 0, \end{cases} \tag{19}$$

where n is a given positive integer. Notice that this is the case where we are prescribing both u and its normal derivative on the boundary.

(a) Use separation of variables to show that the function

$$u(x,y) = \frac{1}{n^2}\frac{e^{ny} - e^{-ny}}{2}\sin(nx) \tag{20}$$

is a solution of (19).

(b) Taking the limit $n \to \infty$ in (19) and (20), what can you conclude about the well-posedness of the boundary value problem (19)?

Solution. Taking the limit in (19) gives the problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ u = 0 & \text{for } y = 0 \\ \frac{\partial u}{\partial y} = 0 & \text{for } y = 0, \end{cases}$$

which has $u = 0$ as solution. On the other hand, taking the limit of the solutions (20), gives

$$u \to \infty$$

for $y > 0$. Thus, the limit of solutions to (19) does not converge a solution of the limit of (19). This shows that solutions do not depend continuously on the initial data, and therefore (19) is not well-posed.

Question 9. Using the Green's function for a ball of radius one,

$$G(x,y) = \Gamma(y-x) - \Gamma\big(|x|(y - \frac{x}{|x|^2})\big),$$

show that if u is a positive function that solves

$$\Delta u = 0, \text{ in } B_R(0),$$

then

$$\frac{R^{n-2}(R-|x|)}{(R+|x|)^{n-1}} u(0) \le u(x) \le \frac{R^{n-2}(R+|x|)}{(R-|x|)^{n-1}} u(0).$$

Solution. Consider first the case $R = 1$, and let g denote u restricted to $\partial B_1(0)$ (it is assumed that u is defined up to the boundary). The representation formula gives

$$u(x) = -\int_{\partial B_1(0)} g(y) \partial_\nu G(x,y)\, ds(y). \tag{21}$$

Since

$$\frac{\partial \Gamma(y-x)}{\partial y_i} = \frac{1}{n\alpha(n)} \frac{x_i - y_i}{|x-y|^n},$$

and

$$\begin{aligned} \frac{\partial \Gamma\big(|x|(y - \frac{x}{|x|^2})\big)}{\partial y_i} &= -\frac{1}{n\alpha(n)} \frac{y_i|x|^2 - x_i}{(|x||y - \frac{x}{|x|^2}|)^n} \\ &= -\frac{1}{n\alpha(n)} \frac{y_i|x|^2 - x_i}{|x-y|^n}, \end{aligned}$$

for $y \in \partial B_1(0)$. Since $\nu_i = y_i$ on $\partial B_1(0)$,

$$\begin{aligned} \partial_\nu G(x,y) &= \sum_{i=1}^n y_i \frac{\partial G(y,x)}{\partial y_i} \\ &= -\frac{1}{n\alpha(n)} \frac{1}{|x-y|^n} \sum_{i=1}^n y_i \big((y_i - x_i) - y_i|x|^2 + x_i\big) \\ &= -\frac{1}{n\alpha(n)} \frac{1 - |x|^2}{|x-y|^n}. \end{aligned} \tag{22}$$

Using (22) into (21) gives

$$u(x) = \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B_1(0)} \frac{g(y)}{|x-y|^n}\, ds(y). \tag{23}$$

Consider now the case of arbitrary $R > 0$. If u solves $\Delta u = 0$ in $B_R(0)$, then the function $\widetilde{u}(x) = u(Rx)$ solves $\Delta \widetilde{u} = 0$ in $B_1(0)$. Changing variables in (23) gives

$$u(x) = \frac{R^2 - |x|^2}{n\alpha(n)R} \int_{\partial B_R(0)} \frac{g(y)}{|x-y|^n}\, ds(y), \tag{24}$$

for $x \in B_R(0)$. We shall now use (24) to solve the problem. For $y \in \partial B_R(0)$, it holds that $|x-y| \leq |x| + R$. Thus (24) and the fact that u is positive imply

$$\begin{aligned} u(x) &= \frac{R^2 - |x|^2}{n\alpha(n)R} \int_{\partial B_R(0)} \frac{g(y)}{|x-y|^n}\, ds(y) \\ &\geq \frac{R^2 - |x|^2}{n\alpha(n)R} \int_{\partial B_R(0)} \frac{g(y)}{(|x|+R)^n}\, ds(y) \\ &= \frac{R^2 - |x|^2}{(|x|+R)^n} \frac{1}{n\alpha(n)R} \int_{\partial B_R(0)} g(y)\, ds(y) \\ &= \frac{R - |x|}{(|x|+R)^{n-1}} \frac{1}{n\alpha(n)R} \int_{\partial B_R(0)} g(y)\, ds(y) \\ &= \frac{R^2 - |x|^2}{(|x|+R)^n} R^{n-2} \frac{1}{n\alpha(n)R^{n-1}} \int_{\partial B_R(0)} g(y)\, ds(y) \\ &= \frac{R^{n-2}(R^2 - |x|^2)}{(|x|+R)^n} u(0), \end{aligned}$$

where in the last step we used the mean value formula:

$$u(0) = \frac{1}{n\alpha(n)R^{n-1}} \int_{\partial B_R(0)} g(y)\, ds(y).$$

The other inequality is similarly proven using that $|x-y| \geq R - |x|$ for $y \in \partial B_R(0)$.

Question 10. Prove that the Green function is symmetric, i.e.,

$$G(x, y) = G(y, x),$$

for all $x, y \in \Omega$, where Ω is the domain of definition of the problem.
Hint: Define $v(z) = G(x, z)$, $w(z) = G(y, z)$, apply Green's identity on the domain $U_\varepsilon = \Omega \backslash (B_\varepsilon(x) \cup B_\varepsilon(y))$, and take the limit $\varepsilon \to 0^+$.
Solution. Let v and w be as above. Then $\Delta v = 0$ for $z \neq x$, $\Delta w = 0$ for $z \neq y$, and $v = 0 = w$ on $\partial\Omega$. Applying Green's identity on U_ε gives

$$\begin{aligned}\int_{U_\varepsilon} (v\Delta w - w\Delta v) &= \int_{\partial U_\varepsilon} (v\partial_\nu w - w\partial_\nu v) \\ &= \int_{\partial\Omega} (v\partial_\nu w - w\partial_\nu v) \\ &+ \int_{\partial B_\varepsilon(x)} (v\partial_{\nu_{in}} w - w\partial_{\nu_{in}} v) + \int_{\partial B_\varepsilon(y)} (v\partial_{\nu_{in}} w - w\partial_{\nu_{in}} v),\end{aligned}$$

where ν_{in} denotes the inner normal. Since $\Delta v = 0$ for $z \neq x$, $\Delta w = 0$ for $z \neq y$, and $v = 0 = w$ on $\partial\Omega$, the above reduces to

$$\int_{\partial B_\varepsilon(x)} (v\partial_{\nu_{in}} w - w\partial_{\nu_{in}} v) + \int_{\partial B_\varepsilon(y)} (v\partial_{\nu_{in}} w - w\partial_{\nu_{in}} v) = 0. \tag{25}$$

Since w is a C^2 function outside $B_\varepsilon(y)$,

$$|\partial_{\nu_{in}} w| \leq \max_{\partial B_\varepsilon(x)} |\nabla w| \leq C,$$

for some constant C. Thus,

$$\begin{aligned}\left| \int_{\partial B_\varepsilon(x)} v\partial_{\nu_{in}} w \right| &\leq C \int_{\partial B_\varepsilon(x)} |v| \\ &\leq C \max_{\partial B_\varepsilon(x)} |v| \int_{\partial B_\varepsilon(x)} ds \\ &\leq C' \varepsilon^{2-n} \varepsilon^{n-1} \\ &= C' \varepsilon,\end{aligned}$$

for some constant C'. Similarly,

$$\left| \int_{\partial B_\varepsilon(y)} w\partial_{\nu_{in}} v \right| \leq C' \varepsilon.$$

Thus, taking the limit $\varepsilon \to 0^+$ in (25),

$$\lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon(x)} w\partial_{\nu_{in}} v = \lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon(y)} v\partial_{\nu_{in}} w. \tag{26}$$

Recall that

$$v(z) = G(x, z) = \Gamma(x - z) + h_x(z),$$

where h_x is a harmonic function in Ω equal to $-\Gamma(x - z)$ on $\partial\Omega$. Since h_x is a C^2 function away from $\partial\Omega$, arguing as above yields

$$\lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon(x)} w\partial_{\nu_{in}} h_x = 0,$$

and the same argument shows that

$$\lim_{\varepsilon\to 0^+}\int_{\partial B_\varepsilon(y)} v\partial_{\nu_{in}} h_z = 0.$$

(26) now reads

$$\lim_{\varepsilon\to 0^+}\int_{\partial B_\varepsilon(x)} w\partial_{\nu_{in}}\Gamma(x-z) = \lim_{\varepsilon\to 0^+}\int_{\partial B_\varepsilon(y)} v\partial_{\nu_{in}}\Gamma(y-z).$$

The above limits were computed in class (see the construction of solutions for the Poisson equation),

$$\lim_{\varepsilon\to 0^+}\int_{\partial B_\varepsilon(x)} w\partial_{\nu_{in}}\Gamma(x-z) = w(x),$$

and

$$\lim_{\varepsilon\to 0^+}\int_{\partial B_\varepsilon(y)} v\partial_{\nu_{in}}\Gamma(y-z) = v(y),$$

which gives the result.

Question 11. Consider the time independent Schrödinger equation studied in class:

$$-\frac{\hbar^2}{2\mu}\Delta\psi + V\psi = E\psi.$$

Show that, under suitable decay conditions on ψ for $|x| \to \infty$, the energy E is always a real number. *Hint:* Similar to when we showed that E is real in the case of the radial equation.

Solution. Multiply the equation by ψ^*, integrate over $B_R(0)$, $R > 0$, and integrate by parts the Laplacian to obtain

$$\frac{\hbar^2}{2\mu}\int_{B_R(0)} |\nabla\psi|^2 - \frac{\hbar^2}{2\mu}\int_{\partial B_R(0)} \psi^*\partial_\nu\psi + \int_{B_R(0)} V|\psi|^2 = E\int_{B_R(0)} |\psi|^2.$$

We want to take the limit $R \to \infty$ and guarantee that the above integrals are finite in the limit. We also want the boundary term to vanish since it is the only integral that is not necessarily real in the above equality.

Recalling that in polar coordinates in $\mathbb{R}^n$

$$\int_{B_R(0)} (\cdots) = \int_{S^{n-1}} \left(\int_0^R (\cdots) r^{n-1}\, dr\right) d\omega,$$

and

$$\int_{\partial B_R(0)} (\cdots) = \int_{S^{n-1}} (\cdots) R^{n-1}\, d\omega,$$

we obtain the desired results if, for instance,

$$|\psi| \le \frac{C}{|x|^{\frac{n}{2}+1}},$$

and

$$|\nabla\psi| \le \frac{C}{|x|^{\frac{n}{2}+2}},$$

provided that V is also a function that decays for large $|x|$.

Question 12. Let $\Omega \subseteq \mathbb{R}^n$.

(a) Show that any integrable function u defines a distribution via

$$\langle u, f\rangle = \int_\Omega uf.$$

(b) Suppose now that $u \in C_c^\infty(\Omega)$. Show that the weak derivative of u, when u is thought of as a distribution (see part (a)), agrees with the usual derivative of u.

Solution. If u is integrable in Ω, i.e., $\int_\Omega |u| < \infty$, then it defines a distribution by

$$\langle u, f\rangle = \int_\Omega u(x)f(x)\,dx, \tag{27}$$

$f \in C_c^\infty(\Omega)$. To see this, first notice that if we let $K \subset \Omega$ be a compact set such that $\text{supp}(f) \subset K$, then

$$\left|\int_\Omega u(x)f(x)\,dx,\right| \leq M\int_K |u(x)|\,dx < \infty,$$

since u is integrable, and where we used that $|f| \leq M$ for some M. Thus, $\langle u, f\rangle$ is well-defined. Linearity follows from linearity of the integral. Finally, if $f_j \to f$ in $C_c^\infty(\Omega)$ and we choose a compact set $K \subset \Omega$ such that $\text{supp}(f_j) \subset K$ for all j (which exists by the definition of convergence in $C_c^\infty(\Omega)$), then

$$\lim_{j\to\infty}\langle u, f_j\rangle = \lim_{j\to\infty}\int_K u(x)f_j(x)\,dx = \int_K u(x)\lim_{j\to\infty} f_j(x)\,dx$$
$$= \int_K u(x)f(x)\,dx = \int_\Omega u(x)f(x)\,dx = \langle u, f_j\rangle.$$

For part (b), let $u \in C_c^\infty(\Omega)$. Then its weak derivative is given by

$$\langle D^\alpha u, f\rangle = (-1)^\alpha\langle u, D^\alpha f\rangle.$$

In view of (27), we can rewrite the right-hand side of this expression as

$$\langle D^\alpha u, f\rangle = (-1)^\alpha\int_\Omega u(x)D^\alpha f(x)\,dx, \tag{28}$$

On the other hand, denote by $\partial^\alpha u$ the ordinary derivative of u, i.e.,

$$\partial^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}.$$

As $\partial^\alpha u \in C_c^\infty(\Omega)$, it also defines a distribution via

$$\langle \partial^\alpha u, f\rangle = \int_\Omega \partial^\alpha u(x)f(x)\,dx.$$

Integrating this expression by parts $|\alpha|$ times gives

$$\langle \partial^\alpha u, f\rangle = (-1)^{|\alpha|}\int_\Omega u(x)D^\alpha f(x)\,dx, \tag{29}$$

where the integration by parts does not yield any integral on the boundary because f is a test function. Subtracting (28) and (29),

$$\langle D^\alpha u - \partial^\alpha u, f\rangle = 0.$$

Since $f \in C_c^\infty(\Omega)$ is arbitrary, this gives $D^\alpha u = \partial^\alpha u$.

Question 13. Consider the function $u : \Omega \to \mathbb{R}$, where $\Omega = (-3, 3) \subset \mathbb{R}$, given by

$$u(x) = \begin{cases} 0, & -3 < x \leq -2, \\ 2x + 4, & -2 < x \leq 0, \\ -x + 1, & 0 < x \leq 1, \\ 0, & 1 < x < 3. \end{cases}$$

Show that u defines a distribution, and that its weak derivative is

$$u' = 2\chi_{[-2,0]} - \chi_{[0,1]} - 3\delta_0,$$

where δ_0 is the Dirac-delta distribution centered at zero, and $\chi_{[a,b]}$ is given by

$$\chi_{[a,b]}(x) = \begin{cases} 1, & x \in [a, b], \\ 0, & x \notin [a, b]. \end{cases}$$

Solution. u is integrable, thus it defines a distribution. Its derivative is given by

$$\langle u', f \rangle = -\langle u, f' \rangle, \tag{30}$$

for all $f \in C_c^\infty(\Omega)$. But

$$\begin{aligned} \langle u, f' \rangle = \int_\Omega u f' &= \int_{-2}^{0} (2x + 4) f'(x)\, dx + \int_0^1 (-x + 1) f'(x)\, dx \\ &= -\int_{-2}^{0} 2f(x)\, dx + (2x + 4) f(x)\Big|_{-2}^{0} - \int_0^1 (-1) f(x)\, dx + (-x + 1) f(x)\Big|_0^1 \\ &= -2 \int_{-2}^{0} f + \int_0^1 f + 3f(0) \\ &= -\int_\Omega (2\chi_{[-2,0]} - \chi_{[0,1]}) f + 3f(0), \end{aligned}$$

Comparing with (30) we conclude that

$$u' = 2\chi_{[-2,0]} - \chi_{[0,1]} - 3\delta_0,$$

where δ_0 is the Dirac-delta distribution centered at zero.

Question 14.

(a) Compute the (weak) derivative of the Dirac-delta function.

(b) Show that any distribution has infinitely many weak derivatives.

Solution. From the definition of weak derivative,

$$\begin{aligned}\langle D^\alpha \delta_x, f\rangle &= (-1)^{|\alpha|}\langle \delta_x, D^\alpha f\rangle \\ &= (-1)^{|\alpha|} D^\alpha f(x).\end{aligned}$$

Thus, $D^\alpha \delta_x$ is the distribution that associates to each test function f, the α-derivative of f evaluated at x (which is a real number).

For part (b), notice that

$$\langle D^\alpha \varphi, f\rangle = (-1)^{|\alpha|}\langle \varphi, D^\alpha f\rangle$$

is well-defined for arbitrary α since $D^\alpha f \in C_c^\infty(\Omega)$ if f does.

Question 15. For $a > 0$, define

$$\phi_a(t) = \begin{cases} \frac{1}{a}, & |t| \leq \frac{a}{2}, \\ 0, & |t| > \frac{a}{2}. \end{cases}$$

(a) Show that

$$\lim_{a\to 0^+} \int_{-\infty}^{\infty} f(t)\phi_a(t)\, dt = \delta(f),$$

for all $f \in C_c^\infty(\mathbb{R})$.

(b) Compute

$$\lim_{a\to 0^+} \phi_a(t).$$

(c) Show that

$$\lim_{a\to 0^+} \int_{-\infty}^{\infty} f(t)\phi_a(t)\, dt \neq \int_{-\infty}^{\infty} f(t) \lim_{a\to 0^+} \phi_a(t)\, dt.$$

Solution. Notice that

$$\int_{-\infty}^{\infty} f(t)\phi_a(t)\, dt = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(t)\, dt,$$

and that

$$\frac{1}{a}\min_{[-\frac{a}{2},\frac{a}{2}]} f(t)\, a \leq \int_{-\frac{a}{2}}^{\frac{a}{2}} f(t)\phi_a(t)\, dt \leq \frac{1}{a}\max_{[-\frac{a}{2},\frac{a}{2}]} f(t)\, a.$$

Combining these last two expressions produces

$$\min_{[-\frac{a}{2},\frac{a}{2}]} f(t) \leq \int_{-\infty}^{\infty} f(t)\phi_a(t)\, dt \leq \max_{[-\frac{a}{2},\frac{a}{2}]} f(t).$$

Since f is continuous, in the limit $a \to 0^+$,

$$\lim_{a\to 0^+} \min_{[-\frac{a}{2},\frac{a}{2}]} f(t) = f(0),$$

and

$$\lim_{a\to 0^+} \max_{[-\frac{a}{2},\frac{a}{2}]} f(t) = f(0),$$

and therefore the squeeze theorem gives

$$\lim_{a\to 0^+} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(t)\phi_a(t)\, dt = f(0) = \delta(f).$$

For part (b), one immediately finds

$$\lim_{a\to 0^+} \phi_a(t) = \begin{cases} 0, & t \neq 0, \\ \infty, & t = 0. \end{cases} \tag{31}$$

For part (c), notice that since

$$\lim_{a\to 0^+} \phi_a(t)$$

is not defined at zero, the integral

$$\int_{-\infty}^{\infty} f(t) \lim_{a\to 0^+} \phi_a(t)\,dt$$

has to be understood as an improper integral, i.e.,

$$\begin{aligned}\int_{-\infty}^{\infty} f(t) \lim_{a\to 0^+} \phi_a(t)\,dt &= \lim_{T\to 0^-} \int_{-\infty}^{T} f(t) \lim_{a\to 0^+} \phi_a(t)\,dt \\ &\quad + \lim_{R\to 0^+} \int_{R}^{\infty} f(t) \lim_{a\to 0^+} \phi_a(t)\,dt.\end{aligned}$$

But in light of (31),

$$\int_{-\infty}^{T} f(t) \lim_{a\to 0^+} \phi_a(t)\,dt = 0,\ T < 0,$$

and

$$\int_{R}^{\infty} f(t) \lim_{a\to 0^+} \phi_a(t)\,dt = 0,\ R > 0,$$

which gives

$$f(0) = \lim_{a\to 0^+} \int_{-\infty}^{\infty} f(t)\phi_a(t)\,dt \neq \int_{-\infty}^{\infty} f(t) \lim_{a\to 0^+} \phi_a(t)\,dt = 0,$$

since f is an arbitrary test function.

Question 16. Define $\varphi : C_c^\infty(\Omega) \to \mathbb{R}$ by

$$\langle \varphi, f \rangle = \begin{cases} 1, & \text{if } f(0) > 0, \\ 0, & \text{if } f(0) \leq 0. \end{cases}$$

Show that φ is not continuous.
Hint: Consider a sequence in $\{f_j\}_{j=1}^\infty \subset C_c^\infty(\Omega)$ such that $f_j(0) > 0$ for all j and $f_j(0) \to 0$.
Solution. Consider the function

$$f(x) = \begin{cases} \exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

Thus, $f(x) > 0$ for $|x| < 1$ and $f(x) = 0$ otherwise. As discussed in class, $f \in C_c^\infty(\Omega)$. Let

$$z_j = (1 - \frac{1}{j}, 0, \ldots, 0), \; j = 1, 2, \ldots$$

and set

$$f_j(x) = f(x - z_j).$$

It follows that $f_j(x) > 0$ on $B_1(x - z_j)$ and $f_j(x) = 0$ otherwise. Because $0 \in B_1(x - z_j)$ for every j, we have $\langle \varphi, f_j \rangle = 1$. On the other hand, $f_j \to g$ in $C_c^\infty(\Omega)$ as $j \to \infty$, where g is given by $g(x) = f(x - e_1)$, $e_1 = (1, 0, \ldots, 0)$. But $g(0) = 0$, thus $\langle \varphi, g \rangle = 0$, which shows that φ is not continuous.

Question 17.

(a) Solve

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in \mathbb{R},\ t > 0, \\ u(0,x) = 0,\ u_t(0,x) = g(x), & x \in \mathbb{R}, \end{cases}$$

where

$$g(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Find $u(\frac{1}{2}, x)$ and $u(t, \frac{1}{2})$.

(c) Let $\phi(x) = u(\frac{1}{2}, x)$ and $\psi(t) = u(t, \frac{1}{2})$. Show that ϕ and ψ are distributions, and compute their second weak derivative.

(d) Use (c) to give an interpretation of the (non-classical) solution that you found in (a).

Solution. Using D'Alembert's formula, we find

$$u(t,x) = \begin{cases} 0, & x + t \le 0, \\ \frac{x+t}{2}, & x - t \le 0,\ 0 \le x + t \le 1, \\ \frac{1}{2}, & x - t \le 0,\ 1 \le x + t, \\ t, & 0 \le x - t \le 1,\ x + t \le 1, \\ \frac{1-x+t}{2}, & 0 \le x - t \le 1, 1 \le\ x + t, \\ 0, & x - t \ge 1. \end{cases}$$

From the above,

$$u(\frac{1}{2}, x) = \begin{cases} 0, & x \le -\frac{1}{2}, \\ \frac{x}{2} + \frac{1}{4}, & -\frac{1}{2} \le x \le \frac{1}{2} \\ \frac{3}{4} - \frac{x}{2}, & \frac{1}{2} \le x \le \frac{3}{2} \\ 0, & \frac{3}{2} \le x, \end{cases}$$

and

$$u(t, \frac{1}{2}) = \begin{cases} 0, & t \le 0, \\ t, & 0 \le t \le \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} \le t, \end{cases}$$

Next, we compute the weak derivatives.

$$\langle \varphi'', f \rangle = \langle \varphi, f'' \rangle,$$

for all $f \in C_c^\infty(\mathbb{R})$. The right-hand side of this expression is given by

$$\begin{aligned}
\langle \varphi, f'' \rangle &= \int_{-\infty}^{\infty} \varphi f'' \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{x}{2} + \frac{1}{4} \right) f''(x)\, dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \left(\frac{3}{4} - \frac{x}{2} \right) f''(x)\, dx \\
&= -\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} f' + \left[\left(\frac{x}{2} + \frac{1}{4} \right) f' \right]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{\frac{1}{2}}^{\frac{3}{2}} \left(-\frac{1}{2} \right) f' + \left[\left(\frac{3}{4} - \frac{x}{2} \right) f' \right]_{\frac{1}{2}}^{\frac{3}{2}} \\
&= -\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} f' + \frac{2}{4} f'(\frac{1}{2}) + \frac{1}{2} \int_{\frac{1}{2}}^{\frac{3}{2}} f' - \frac{2}{4} f'(\frac{1}{2}) \\
&= -\frac{1}{2} f \Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \frac{1}{2} f \Big|_{\frac{1}{2}}^{\frac{3}{2}} \\
&= -f(\frac{1}{2}) + \frac{1}{2} \left(f(\frac{3}{2}) + f(-\frac{1}{2}) \right).
\end{aligned}$$

A similar argument yields

$$\langle \psi'', f \rangle = -f(\frac{1}{2}) + f(0).$$

We interpret these calculations as follows. The solution $u(t, x)$ found above is not classical, thus we cannot plug it in directly in the wave equation and evaluate it at the points where derivatives are not defined, such as $(\frac{1}{2}, \frac{1}{2})$. If we interpret the derivatives as weak derivatives though, then we can imagine smearing out the solution near $(\frac{1}{2}, \frac{1}{2})$, i.e., we can choose test functions that are supported in a very small neighborhood of $(\frac{1}{2}, \frac{1}{2})$. In this situation, we can *heuristically* think of the point-wise expressions

$$u_{tt}(\frac{1}{2}, \frac{1}{2}) = \langle \psi'', f \rangle = -f(\frac{1}{2}) = -\langle \delta_{\frac{1}{2}}, f \rangle,$$

and

$$u_{xx}(\frac{1}{2}, \frac{1}{2}) = \langle \varphi'', f \rangle = -f(\frac{1}{2}) = -\langle \delta_{\frac{1}{2}}, f \rangle,$$

so that $u_{tt}(\frac{1}{2}, \frac{1}{2})$“ = ” $- \delta_{\frac{1}{2}}$ and $u_{xx}(\frac{1}{2}, \frac{1}{2})$“ = ” $- \delta_{\frac{1}{2}}$, giving a “solution” satisfying $u_{tt}(\frac{1}{2}, \frac{1}{2}) - u_{xx}(\frac{1}{2}, \frac{1}{2}) = 0$.