VANDERBILT UNIVERSITY
MATH 234 — INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS
PRACTICE FINAL.

Question 1. Solve the following initial-boundary value problem

$$\begin{cases} u_{sr} = 0, & s \geq -r, -\infty < r < \infty, \\ u(-r,r) = F(r), & -\infty < r < \infty, \\ u_s(-r,r) = G(r), & -\infty < r < \infty, \end{cases}$$

where $u = u(s,r)$ is the unknown, and F and G are given C^∞ functions.
Hint: Change variables $t = s + r$, $x = s - r$.

Question 2. Let $u(t, x)$ be a solution to the following initial-value problem:

$$\begin{cases} u_{tt} - u_{xx} = f(t, x), & -\infty < x < \infty,\ t > 0, \\ u(0, x) = g(x),\ u_t(0, x) = h(x), & -\infty < x < \infty, \end{cases}$$

where f, g, and h are C^∞ functions. Assume that there exist numbers X, Y, and Z, such that

$$|f(t, x)| \leq X,\ |g(x)| \leq Y,\ |h(x)| \leq Z,$$

for all $t \geq 0$, $x \in \mathbb{R}$. Show that for any $t > 0$, and any $x \in \mathbb{R}$, it holds that

$$|u(t, x)| \leq Y + tZ + \frac{1}{2} X t^2.$$

Hint: D'Alembert and Duhamel.

Question 3. Let v and w be, respectively, solutions to

$$\begin{cases} v_{tt} - \Delta v = 0, & x \in \mathbb{R}^n,\ t > 0, \\ v(0,x) = f_1(x),\ v_t(0,x) = g_1(x), & x \in \mathbb{R}^n, \end{cases}$$

and

$$\begin{cases} w_{tt} - \Delta w = 0, & x \in \mathbb{R}^n,\ t > 0, \\ w(0,x) = f_2(x),\ w_t(0,x) = g_2(x), & x \in \mathbb{R}^n, \end{cases}$$

where f_1, f_2, g_1, and g_2 are given smooth functions. Suppose that $f_1(x) = f_2(x)$ for all $x \in B_1(0)$, and $g_1(x) = g_2(x)$ for all $x \in B_1(0)$. Show that $v(t,x) = w(t,x)$ for all $(t,x) \in C$, where C is the cone

$$C = \left\{ (t,x) \in [0,\infty) \times \mathbb{R}^n \,\middle|\, 0 \le t \le 1,\ |x| \le 1 - t \right\}.$$

Hint: Use

$$E(t) = \frac{1}{2} \int_{B_{1-t}(0)} \left[(\partial_t u(t,x))^2 + c^2 |\nabla u(t,x)|^2 \right] dx,$$

where ∇ is the gradient in $\mathbb{R}^n$ and $|\nabla u(t,x)|$ is the norm of the vector $\nabla u(t,x)$.

Question 5. Prove the following improved version of the maximum principle. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. If $u \in C^2(\overline{\Omega})$ satisfies

$$\Delta u \geq 0 \text{ in } \Omega,$$

then u attains its maximum on the boundary, i.e.,

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Hint: Assume first that $\Delta u > 0$, and show that this cannot happen if u has a local maximum in Ω. For the case $\Delta u \geq 0$, set $u_\varepsilon = u + \varepsilon e^{x_1}$, where $\varepsilon > 0$, and conclude that $\Delta u_\varepsilon > 0$. Obtain the result by taking the limit $\varepsilon \to 0^+$.

Formulate, and prove, a similar statement for the minimum of u.

Question 6. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the Dirichlet problem for the Laplacian

$$\begin{cases} \Delta u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $f : \overline{\Omega} \to \mathbb{R}$, and $g : \partial\Omega \to \mathbb{R}$ are smooth (i.e., C^∞) functions. Show that this problem is well-posed.

Hint: For existence, you can simply quote the results from class. For continuous dependence on the parameters, use uniqueness to show that one can write $u = v + w$, where v solves (1) with $g = 0$, and w solves (1) with $f = 0$. Next, define the functions

$$(e^{2\alpha d} - e^{\alpha(x_1+d)}) \max_{\overline{\Omega}} |f|,$$

and

$$v_- = -(e^{2\beta d} - e^{\beta(x_1+d)}) \max_{\overline{\Omega}} |f|.$$

Show that for suitable choice of the constants α, β, and d, one can apply the result of question 5 to the functions $v_+ - v$ and $v_- - v$ to conclude that

$$|v| \le C \max_{\overline{\Omega}} |f|, \tag{2}$$

for some constant C depending on α, β, and d. Finally, use (2) to conclude that u depends continuously on the data of the problem.

Question 7. Prove the result of the previous question using the Green function.

Question 8. Let $\mathbb{R}^2_+$ be the upper half plane in $\mathbb{R}^2$, i.e.,

$$\mathbb{R}^2_+ = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, y > 0 \right\}.$$

Consider the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ u = 0 & \text{for } y = 0 \\ \frac{\partial u}{\partial y} = \frac{1}{n} \sin(nx) & \text{for } y = 0, \end{cases} \tag{3}$$

where n is a given positive integer. Notice that this is the case where we are prescribing both u and its normal derivative on the boundary.

(a) Use separation of variables to show that the function

$$u(x, y) = \frac{1}{n^2} \frac{e^{ny} - e^{-ny}}{2} \sin(nx) \tag{4}$$

is a solution of (3).

(b) Taking the limit $n \to \infty$ in (3) and (4), what can you conclude about the well-posedness of the boundary value problem (3)?

Question 9. Using the Green's function for a ball of radius one,

$$G(x,y) = \Gamma(y-x) - \Gamma\big(|x|(y - \frac{x}{|x|^2})\big),$$

show that if u is a positive function that solves

$$\Delta u = 0, \text{ in } B_R(0),$$

then

$$\frac{R^{n-2}(R-|x|)}{(R+|x|)^{n-1}}u(0) \le u(x) \le \frac{R^{n-2}(R+|x|)}{(R-|x|)^{n-1}}u(0).$$

Question 10. Prove that the Green function is symmetric, i.e.,

$$G(x,y) = G(y,x),$$

for all $x, y \in \Omega$, where Ω is the domain of definition of the problem.
Hint: Define $v(z) = G(x,z)$, $w(z) = G(y,z)$, apply Green's identity on the domain $U_\varepsilon = \Omega \backslash (B_\varepsilon(x) \cup B_\varepsilon(y))$, and take the limit $\varepsilon \to 0^+$.

Question 11. Consider the time independent Schrödinger equation studied in class:

$$-\frac{\hbar^2}{2\mu}\Delta\psi + V\psi = E\psi.$$

Show that, under suitable decay conditions on ψ for $|x| \to \infty$, the energy E is always a real number.
Hint: Similar to when we showed that E is real in the case of the radial equation.

Question 12. Let $\Omega \subseteq \mathbb{R}^n$.

(a) Show that any integrable function u defines a distribution via

$$\langle u, f \rangle = \int_\Omega uf.$$

(b) Suppose now that $u \in C_c^\infty(\Omega)$. Show that the weak derivative of u, when u is thought of as a distribution (see part (a)), agrees with the usual derivative of u.

Question 13. Consider the function $u : \Omega \to \mathbb{R}$, where $\Omega = (-3, 3) \subset \mathbb{R}$, given by

$$u(x) = \begin{cases} 0, & -3 < x \leq -2, \\ 2x + 4, & -2 < x \leq 0, \\ -x + 1, & 0 < x \leq 1, \\ 0, & 1 < x < 3. \end{cases}$$

Show that u defines a distribution, and that its weak derivative is

$$u' = 2\chi_{[-2,0]} - \chi_{[0,1]} - 3\delta_0,$$

where δ_0 is the Dirac-delta distribution centered at zero, and $\chi_{[a,b]}$ is given by

$$\chi_{[a,b]}(x) = \begin{cases} 1, & x \in [a, b], \\ 0, & x \notin [a, b]. \end{cases}$$

Question 14.

(a) Compute the (weak) derivative of the Dirac-delta function.

(b) Show that any distribution has infinitely many weak derivatives.

Question 15. For $a > 0$, define

$$\phi_a(t) = \begin{cases} \frac{1}{a}, & |t| \leq \frac{a}{2}, \\ 0, & |t| > \frac{a}{2}. \end{cases}$$

(a) Show that

$$\lim_{a \to 0^+} \int_{-\infty}^{\infty} f(t)\phi_a(t)\, dt = \delta(f),$$

for all $f \in C_c^\infty(\mathbb{R})$.

(b) Compute

$$\lim_{a \to 0^+} \phi_a(t).$$

(c) Show that

$$\lim_{a \to 0^+} \int_{-\infty}^{\infty} f(t)\phi_a(t)\, dt \neq \int_{-\infty}^{\infty} f(t) \lim_{a \to 0^+} \phi_a(t)\, dt.$$

Question 16. Define $\varphi : C_c^\infty(\Omega) \to \mathbb{R}$ by

$$\langle \varphi, f \rangle = \begin{cases} 1, & \text{if } f(0) > 0, \\ 0, & \text{if } f(0) \leq 0. \end{cases}$$

Show that φ is not continuous.
Hint: Consider a sequence in $\{f_j\}_{j=1}^\infty \subset C_c^\infty(\Omega)$ such that $f_j(0) > 0$ for all j and $f_j(0) \to 0$.

Question 17.

(a) Solve

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in \mathbb{R},\ t > 0, \\ u(0,x) = 0,\ u_t(0,x) = g(x), & x \in \mathbb{R}, \end{cases}$$

where

$$g(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Find $u(\frac{1}{2}, x)$ and $u(t, \frac{1}{2})$.

(c) Let $\phi(x) = u(\frac{1}{2}, x)$ and $\psi(t) = u(t, \frac{1}{2})$. Show that ϕ and ψ are distributions, and compute their second weak derivative.

(d) Use (c) to give an interpretation of the (non-classical) solution that you found in (a).