VANDERBILT UNIVERSITY, MATH 234 SPRING 14: THE POISSON EQUATION IN $\mathbb{R}^n$.

This is a mix of class notes and homework assignment, whose goal is to solve

$$-\Delta u = f \tag{1}$$

in $\mathbb{R}^n$.

From now on, it is assumed that $n \geq 3$. Let $f \in C^2(\mathbb{R}^n)$, and assume that f has compact support. Recall that in class we defined $\Phi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ by

$$\Phi(x) = \frac{1}{n(n-2)\alpha_n} \frac{1}{|x|^{n-2}}, \tag{2}$$

where α_n is the volume of the unit ball in $\mathbb{R}^n$.

Problem 1. Compute

$$\partial_i |x|,$$

and use this to show that there exists a constant $C > 0$ such that

$$|\partial_i \Phi| \leq \frac{C}{|x|^{n-1}},\, |\partial_{ij}\Phi| \leq \frac{C}{|x|^n},\, i,j = 1,\ldots,n,\, x \neq 0.$$

Solution. Since

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

It follows at once that

$$\partial_i |x| = \frac{x_i}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}} = \frac{x_i}{|x|}.$$

Now use the chain rule to compute

$$\partial_i |x|^{2-n} = (2-n)|x|^{1-n} \frac{x_i}{|x|}.$$

Noticing that

$$\left| \frac{x_i}{|x|} \right| \leq 1,$$

one obtains

$$\left| \partial_i |x|^{2-n|} \right| \leq \frac{n-2}{|x|^{1-n}},$$

from which the first desired inequality follows. Next, use the chain rule again to compute

$$\begin{aligned}
\partial_{ji}|x|^{2-n} &= \partial_j\left((2-n)|x|^{1-n}\frac{x_i}{|x|}\right)\\
&= (2-n)\partial_j\left(\frac{x_i}{|x|^n}\right)\\
&= (2-n)\frac{|x|^n\delta_{ij} - n|x|^{n-1}\frac{x_ix_j}{|x|}}{|x|^{2n}}\\
&= \frac{2-n}{|x|^n}\left(\delta_{ij} - \frac{nx_ix_j}{|x|^2}\right),
\end{aligned}$$

where we used that

$$\partial_j x_i = \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

The second desired inequality now follows from noticing that

$$\left|\delta_{ij} - \frac{nx_ix_j}{|x|^2}\right| \leq 1 + n,$$

since

$$\left|\frac{x_ix_j}{|x|^2}\right| \leq 1.$$

Define $u : \mathbb{R}^n \to \mathbb{R}$ by

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy.$$

Recall that this can also be written as

$$u(x) = \int_{\mathbb{R}^n} \Phi(y)f(x-y)\,dy. \tag{3}$$

In class, we showed that u is well-defined, and that the second derivatives of u exist and satisfy

$$u_{x_ix_j}(x) = \int_{\mathbb{R}^n} \Phi(y)f_{x_ix_j}(x-y)\,dy.$$

Problem 2. Show that $u_{x_ix_j}$ is continuous. Recalling the definition of continuity, you have to show that, given $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists a $\delta > 0$, such that if $|x - x_0| < \delta$, then $|u_{x_ix_j}(x) - u_{x_ix_j}(x_0)| < \varepsilon$. Do this as follows. Fix $\varepsilon > 0$. Write

$$\begin{aligned}
|u_{x_ix_j}(x) - u_{x_ix_j}(x_0)| &= \left|\int_{\mathbb{R}^n} \Phi(y)f_{x_ix_j}(x-y)\,dy - \int_{\mathbb{R}^n} \Phi(y)f_{x_ix_j}(x_0-y)\,dy\right|\\
&= \left|\int_{\mathbb{R}^n} \Phi(y)(f_{x_ix_j}(x-y) - f_{x_ix_j}(x_0-y))\,dy\right|
\end{aligned}$$

Use the continuity of $f_{x_ix_j}$, and the fact that f has compact support (i.e., that $\mathrm{supp}(f) \subset B_R(0)$ for some $R > 0$), to show that given $\varepsilon' > 0$, we can choose $\delta > 0$ so that

$$\begin{aligned}
|u_{x_ix_j}(x) - u_{x_ix_j}(x_0)| &\leq \int_{\mathbb{R}^n} \Phi(y)|f_{x_ix_j}(x-y) - f_{x_ix_j}(x_0-y)|\,dy\\
&\leq \varepsilon' \int_{B_R(0)} \Phi(y)\,dy,
\end{aligned}$$

provided that $|x - x_0| < \delta$. Next, use the expression (2), and integration in polar coordinates (in n dimensions), to show that ε' can be chosen so that

$$\varepsilon' \int_{B_R(0)} \Phi(y)\, dy, < \varepsilon,$$

as desired.

Solution. By continuity, given ε', there exists a $\delta > 0$ such that

$$|f_{x_i x_j}(x - y) - f_{x_i x_j}(x_0 - y)| < \varepsilon',$$

provided that

$$|x - y - (x_0 - y)| = |x - x_0| < \delta.$$

δ may in principle depend on y, but since the support of f (and hence of $f_{x_i x_j}$) is compact, δ can be chosen uniformly. Since ε' can be chosen as small as we want, we set

$$\varepsilon' = \left(\int_{B_R(0)} \Phi(y)\, dy \right)^{-1} \varepsilon.$$

We will now show that (1) holds. The argument here will be slightly simpler than what we did in class, although conceptually it is the same.

From (3), compute

$$\Delta u(x) = \int_{\mathbb{R}^n} \Phi(y) \Delta_x f(x - y)\, dy.$$

Fix $\varepsilon > 0$, and write

$$\Delta u(x) = \int_{B_\varepsilon(0)} \Phi(y) \Delta_x f(x - y)\, dy + \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y) \Delta_x f(x - y)\, dy. \tag{4}$$

Since $\Delta_x f$ is a continuous function and f has compact support, it follows that there exists a constant $M > 0$ such that

$$|\Delta f(x)| \leq M \text{ for all } x \in \mathbb{R}^n.$$

Thus

$$\Big| \int_{B_\varepsilon(0)} \Phi(y) \Delta_x f(x - y)\, dy \Big| \leq M \int_{B_\varepsilon(0)} \Phi(y)\, dy.$$

Problem 3. Using polar coordinates, as done in class, estimate the integral on the right-hand side of the previous expression and show that

$$\lim_{\varepsilon \to 0} \int_{B_\varepsilon(0)} \Phi(y) \Delta_x f(x - y)\, dy = 0.$$

Solution. It follows from

$$0 \leq \int_{B_\varepsilon(0)} \Phi(y)\, dy = \frac{1}{n(n-2)\alpha_n} \int_0^\varepsilon \int_{S^{n-1}} \frac{1}{r^{n-2}} r^{n-1}\, dr d\omega \leq C\varepsilon^2.$$

Problem 4. As done in class, use the chain rule to show that

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y) \Delta_x f(x - y)\, dy = \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y) \Delta_y f(x - y)\, dy.$$

Solution. By the chain rule,

$$\frac{\partial}{\partial y_i}\left(f(x-y)\right)=\sum_{j=1}^{n}\partial_j f(x-y)\frac{\partial(x_j-y_j)}{\partial y_i}=-\partial_i f(x-y),$$

so that

$$\frac{\partial^2}{\partial y_i^2}\left(f(x-y)\right)=\partial_{ii}f(x-y).$$

Thus (4) becomes

$$\Delta u(x)=I+II,$$

where

$$I=\int_{B_\varepsilon(0)}\Phi(y)\Delta_x f(x-y)\,dy.$$

and

$$II=\int_{\mathbb{R}^n\setminus B_\varepsilon(0)}\Phi(y)\Delta_y f(x-y)\,dy.$$

Integrating by parts, II becomes

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_\varepsilon(0)}\Phi(y)\Delta_y f(x-y)\,dy=&-\int_{\mathbb{R}^n\setminus B_\varepsilon(0)}\langle\nabla\Phi(y),\nabla_y f(x-y)\rangle\,dy\\&+\int_{\partial(\mathbb{R}^n\setminus B_\varepsilon(0))}\Phi(y)\partial_\nu f(x-y)ds(y).\end{aligned}$$

Problem 5. Arguing similarly to problem 3, show that

$$\lim_{\varepsilon\to0}\int_{\partial(\mathbb{R}^n\setminus B_\varepsilon(0))}\Phi(y)\partial_\nu f(x-y)ds(y)=0.$$

Solution. Using continuity of the first derivatives of f and the fact that it has compact support, we can find $M>0$ such that

$$|\nabla f|\le M.$$

In particular,

$$|\partial_\nu f|\le M.$$

Thus

$$\begin{aligned}0\le\left|\int_{\partial(\mathbb{R}^n\setminus B_\varepsilon(0))}\Phi(y)\partial_\nu f(x-y)ds(y)\right|&\le M\int_{\partial B_\varepsilon(0)}\Phi(y)\,ds(y)\\&=\frac{M}{n(n-2)\alpha_n}\int_{S^{n-1}}\frac{1}{\varepsilon^{n-2}}\varepsilon^{n-1}\,d\omega\le C\varepsilon,\end{aligned}$$

which gives the result.

Integrating by parts again, II can still be written as

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_\varepsilon(0)}\Phi(y)\Delta_y f(x-y)\,dy=&\int_{\mathbb{R}^n\setminus B_\varepsilon(0)}\Delta\Phi(y)f(x-y)\,dy\\&-\int_{\partial(\mathbb{R}^n\setminus B_\varepsilon(0))}\partial_\nu\Phi(y)f(x-y)ds(y),+III\end{aligned}\tag{5}$$

where

$$III = \int_{\partial(\mathbb{R}^n \backslash B_\varepsilon(0))} \Phi(y) \partial_\nu f(x-y) ds(y).$$

Since Φ satisfies $\Delta \Phi = 0$ in $\mathbb{R}^n \backslash \{0\}$, the first integral on the right-hand side of (5) vanishes. Combining the above calculations then gives

$$\Delta u(x) = I - \int_{\partial(\mathbb{R}^n \backslash B_\varepsilon(0))} \partial_\nu \Phi(y) f(x-y) ds(y) + III. \tag{6}$$

Using (2) and problem 1, we can compute

$$\partial_i \Phi(y) = -\frac{1}{n\alpha_n} \frac{y_i}{|y|^n} = -\frac{1}{n\alpha_n |y|^{n-1}} \frac{y_i}{|y|}.$$

On $\partial(\mathbb{R}^n \backslash B_\varepsilon(0))$, we have $|y| = \varepsilon$ and $\frac{y_i}{|y|} = -\nu_i$ (recall that the negative sign appears because ν is the outer normal to $\partial(\mathbb{R}^n \backslash B_\varepsilon(0))$, which is opposite to the normal to $B_\varepsilon(0)$). Thus, the above becomes

$$\partial_i \Phi(y) = \frac{1}{n\alpha_n \varepsilon^{n-1}} \nu_i,$$

and then

$$\begin{aligned}
\partial_\nu \Phi(y) &= \sum_{i=1}^n \partial_i \Phi(y) \nu_i \\
&= \sum_{i=1}^n \frac{1}{n\alpha_n \varepsilon^{n-1}} \nu_i^2 \\
&= \frac{1}{n\alpha_n \varepsilon^{n-1}} \sum_{i=1}^n \nu_i^2 \\
&= \frac{1}{n\alpha_n \varepsilon^{n-1}} |\nu|^2 \\
&= \frac{1}{n\alpha_n \varepsilon^{n-1}},
\end{aligned}$$

since $|\nu| = 1$. Using this into (6) gives

$$\Delta u(x) = I - \frac{1}{n\alpha_n \varepsilon^{n-1}} \int_{\partial(\mathbb{R}^n \backslash B_\varepsilon(0))} f(x-y) ds(y) + III.$$

Taking the limit $\varepsilon \to 0$, using problems 3 and 5, and noticing that $\Delta u(x)$ does not depend on ε, we obtain

$$\Delta u(x) = -\lim_{\varepsilon \to 0} \frac{1}{n\alpha_n \varepsilon^{n-1}} \int_{\partial(\mathbb{R}^n \backslash B_\varepsilon(0))} f(x-y) ds(y).$$

Notice that, as sets,

$$\partial(\mathbb{R}^n \backslash B_\varepsilon(0)) = \partial B_\varepsilon(0),$$

thus

$$\Delta u(x) = -\lim_{\varepsilon \to 0} \frac{1}{n\alpha_n \varepsilon^{n-1}} \int_{\partial B_\varepsilon(0)} f(x-y) ds(y).$$

Problem 6. Show that, upon changing variables, this last expression becomes

$$\Delta u(x) = -\lim_{\varepsilon\to 0} \frac{1}{n\alpha_n \varepsilon^{n-1}} \int_{\partial B_\varepsilon(x)} f(y).$$

Solution. Set $z = x - y$ and notice that the Jacobian of this change of variables is equal to one.

Notice that $n\alpha_n\varepsilon^{n-1}$ is the volume of $\partial B_\varepsilon(x)$,

$$\Delta u(x) = -\lim_{\varepsilon\to 0} \frac{1}{\mathrm{vol}(\partial B_\varepsilon(x))} \int_{\partial B_\varepsilon(x)} f(y).$$

The right-hand side is the average of f over $\partial B_\varepsilon(x)$. But if we average f over ever smaller concentric spheres, the value of the average approaches the value of f at the center. Hence,

$$\lim_{\varepsilon\to 0} \frac{1}{\mathrm{vol}(\partial B_\varepsilon(x))} \int_{\partial B_\varepsilon(x)} f(y) = f(x),$$

finishing the proof.

Problem 7. Solve Poisson's equation, as above, in the case $n = 2$. *Hint:* chapter 8 of the textbook.

Solution. This is done in chapter 8 of the textbook, see corollary 8.2.

URL: `http://www.disconzi.net/Teaching/MAT234-Spring-14/MAT234-Spring-14.html`