VANDERBILT UNIVERSITY
MATH 234 — INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS
SPRING 14.

Question 1. Give an example of a function that is differentiable but not C^1 (recall that a function is C^1 if all its first derivatives exists and are continuous).

Question 2. Prove *Duhamel's principle:* for $s \geq 0$, let $v(x,t,s)$ be the solution of the following initial value problem (which depends on the parameter s):

$$v_{tt} - c^2 v_{xx} = 0,\ x \in \mathbb{R},\ t > 0,$$
$$v(x,0,s) = 0,\ v_t(x,0,s) = f(x,s),\ x \in \mathbb{R}.$$

Then the function

$$u(x,t) = \int_0^t v(x, t-s, s)\, ds$$

solves

$$u_{tt} - c^2 u_{xx} = f(x,t),\ x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = 0,\ u_t(x,0) = 0,\ x \in \mathbb{R}.$$

Question 3. We say that a domain $\Omega \subset \mathbb{R}^n$ is *bounded* if there exists an $R > 0$ such that $\Omega \subset B_R(0)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Prove the integration by parts formula:

$$\int_\Omega u \partial_i v = -\int_\Omega \partial_i u v + \int_{\partial\Omega} u v \nu_i, \quad i = 1, \dots, n,$$

where ν_i is the i^{th} component of the unit normal to $\partial\Omega$, and $u, v : \overline{\Omega} \to \mathbb{R}$ are sufficiently differentiable functions.

Question 4. Show that the integration by parts formula still holds when Ω is one of the following domains, identifying appropriate conditions on the functions u and v.

(a) The half-ball,

$$\Omega = B_r(0) \bigcap \Big\{ x \in \mathbb{R}^n \,\Big|\, x_n > 0 \Big\}.$$

(b) The cube of size L,

$$\Omega = \Big\{ x \in \mathbb{R}^n \,\Big|\, -\frac{L}{2} < x_i < \frac{L}{2},\ i = 1, \dots, n \Big\}.$$

(c) The cone of vertex z,

$$K(z) = \Big\{ x \in \mathbb{R}^n \,\Big|\, 0 \leq x_n \leq z_n,\ |x' - z'| \leq z_n - x_n \Big\},$$

where $x' = (x_1, x_2, \dots, x_{n-1}, 0)$ and $z' = (z_1, z_2, \dots, z_{n-1}, 0)$.