VANDERBILT UNIVERSITY
MATH 234 — INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS
SPRING 14.

Before doing this assignment, you should make sure you understand the proof of uniqueness of solutions for the one-dimensional wave equation presented in class. This is also covered on section 5.5 of the textbook.

The goal of this assignment is to show uniqueness of solutions for suitable initial-boundary value problems for the wave equation in more than one spatial dimension. In order to do so you will have to use techniques from multivariable calculus, therefore you should brush up on those if necessary.

Recall that in class we looked at the following initial-boundary value problem for the one-dimensional wave equation,

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(t,x), & 0 < x < L,\, t > 0, \\ u_x(t,0) = a(t),\, u_x(t,L) = b(t), & t \geq 0, \\ u(0,x) = f(x),\, u_t(0,x) = g(x), & 0 \leq x \leq L. \end{cases} \tag{1}$$

Let Ω be a bounded domain in $\mathbb{R}^n$ and denote by $\partial\Omega$ its boundary, assumed to be smooth. For concreteness you can think that Ω is the the ball of radius one centered at the origin. Then, for example, in $n = 3$ we have

$$\Omega = \Big\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \,\Big|\, x_1^2 + x_2^2 + x_3^2 < 1 \Big\},$$

and

$$\partial\Omega = \Big\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \,\Big|\, x_1^2 + x_2^2 + x_3^2 = 1 \Big\}.$$

Consider the following initial-boundary value problem for the wave equation in n spatial dimensions,

$$\begin{cases} u_{tt} - c^2 \Delta u = F(t,x), & x \in \Omega,\, t > 0, \\ u(t,x) = h(t,x), & x \in \partial\Omega,\, t \geq 0, \\ u(0,x) = f(x),\, u_t(0,x) = g(x), & x \in \Omega. \end{cases} \tag{2}$$

Notice that here x denotes a point in n-dimensions, i.e., $x = (x_1, x_2, \dots, x_n)$. In particular, we avoid the notation $\vec{x}$.

Question 1. Explain why problem (2) is a higher dimensional analogue of (1). Notice, however, that in (1), it is the derivative of u which is prescribed on the boundary of the interval $[0, L]$, i.e., at $x = 0$ and $x = L$, whereas in (2), it is rather the values of u itself (and not of its derivative) that are given on the boundary $\partial\Omega$. Explain how (1) has to be modified so that it looks more like (2), and reciprocally how (2) would have to be changed in order to look more like (1). (*hint:* in class we drew a picture with a "rectangle" in x and t coordinates, where we indicated the values of $u(0,x)$, $u_x(t,L)$ etc; drawing a similar picture in higher dimensions is useful in this problem. For the sake of drawing the picture, of course, you can assume that $n = 2$ and that Ω is a ball).

Question 2. Suppose now that $n = 2$. Define the energy

$$E(t) = \frac{1}{2} \int_{\Omega} \left[(\partial_t u(t,x))^2 + c^2 |\nabla u(t,x)|^2 \right] \, dx,$$

where ∇ is the gradient in $\mathbb{R}^2$ and $|\nabla u(t,x)|$ is the norm of the vector $\nabla u(t,x)$ (multivariable calculus!). Using the above energy, and an argument similar to the one employed in class for problem (1), show uniqueness of solutions to (2).

Question 3. Repeat question 2 in $n = 3$ dimensions. What can you say about the general case of arbitrary n?

Question 4. In class, we showed the so-called "finite propagation speed" property of the one-dimensional wave equation, which included the definition of a domain of dependence region. Formulate a similar statement for the problem (2). What is the domain of dependence in this case? Illustrate with a picture. Notice that you are not required to demonstrate the finite propagation speed property here, only state it, indicating the relevant domain of dependence.