**VANDERBILT UNIVERSITY**
**MATH 234 — INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS**
**SPRING 14.**

**Question 1.** For each set $U$ below: (i) describe $U$ in words (e.g., the first quadrant, intersection of the ball of radius one with the third quadrant, etc), drawing a picture when possible; (ii) identify $\partial U$, $\mathring{U}$, and $\overline{U}$; (iii) identify the area element of the boundary, i.e., $ds$; (iv) identify the normal to the boundary.

(a)

$$U = \left\{ x \in \mathbb{R}^2 \,\middle|\, |x| < 5 \right\}.$$

(b)

$$U = \left\{ x \in \mathbb{R}^2 \,\middle|\, -2 < x_1 < 2, -1 < x_2 < 1 \right\}.$$

(c)

$$U = \left\{ x \in \mathbb{R}^3 \,\middle|\, -1 < x_1 < 1, -1 < x_2 < 1, -1 < x_3 < 1 \right\}.$$

(d)

$$U = \left\{ x \in \mathbb{R}^2 \,\middle|\, -1 < x_1 < 1, -1 < x_2 < 1 \right\} \bigcap B_1(0).$$

(e)

$$U = \left\{ x \in \mathbb{R}^3 \,\middle|\, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \right\}.$$

(f)

$$U = \left\{ x \in \mathbb{R}^3 \,\middle|\, x_3 > 0 \right\} \bigcap B_r(0).$$

(g)

$$U = \left\{ x \in \mathbb{R}^3 \,\middle|\, \langle x, (1,1,1) \rangle = 0 \right\} \bigcap B_1(0).$$

**Question 2.** Consider the functions below.

$$
\begin{aligned}
(i) \quad & f : \mathbb{R} \to \mathbb{R}, \\
& f(x) = x^3. \\
(ii) \quad & g : \mathbb{R}^2 \to \mathbb{R}, \\
& g(x,y) = xy - x^2 \sin y. \\
(iii) \quad & h : \mathbb{R}^3 \to \mathbb{R}^2, \\
& h(x_1, x_2, x_3) = (x_1 x^2, x_3^2 - x_3 x_1). \\
(iv) \quad & w : \mathbb{R}^3 \to \mathbb{R}^3, \\
& w(x_1, x_2, x_3) = (x_2^2 x_3^2, x_1^2 x_3^2, x_1^2 x_2^2). \\
(v) \quad & z : \mathbb{R}^3 \to \mathbb{R}, \\
& w(x_1, x_2, x_3) = x_1 x_2 x_3
\end{aligned}
$$

Compute:

(a) $\partial_i(f \circ g)$, $i = 1, 2$.

(b) $\nabla g$.

(c) $\nabla(f \circ |h|)$.

(d) $Dw$.

(e) $D(h \circ w)$.

(f) $\partial_i(z \circ w)$, $i = 1, 2, 3$.

For questions 3-7, let $\Omega$ be a domain in $\mathbb{R}^n$, i.e., $\Omega \subseteq \mathbb{R}^n$ is an open and connected set contained in $\mathbb{R}^n$. For concreteness you can imagine that $\Omega$ is the ball of radius one centered at the origin.

This assignment involves concepts that you learned in previous courses (such as vector spaces, etc). If you do not remember them, this is a good chance to refresh your memory since we will be needing some of these concepts later on in the course.

We say that a function $u$ is $k$-times continuously differentiable if all derivatives up to order $k$ of $u$ exist and are continuous (recall that we saw in class an example of a function whose second derivative exists but is not continuous).

Remember that we defined the spaces

$$C^k(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \,\Big|\, u \text{ is } k\text{-times continuously differentiable } \Big\}.$$

**Question 3.** Show that $C^k(\Omega)$ is a vector space.

**Question 4.** Show that the Laplacian $\Delta$ is a linear map between $C^k(\Omega)$ and $C^{k-2}(\Omega)$, $k \geq 2$.

Recall that

$$C^\infty(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \,\Big|\, u \in C^k(\Omega) \text{ for every } k \Big\}.$$

**Question 5.** Show that $C^\infty(\Omega)$ is a vector space.

**Question 6.** Show that the Laplacian $\Delta$ is a linear map from $C^\infty(\Omega)$ to itself.

**Question 7.** Give a reasonable argument for why $C^k(\Omega)$ is an infinite-dimensional vector space. You are not asked to provide a mathematical and rigorous proof. Instead, you should use your knowledge of calculus and linear algebra, as well as the material we learned in class, to construct a sensible explanation, even if only an intuitive one.