VANDERBILT UNIVERSITY
MATH 234 — INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS
LIST OF FORMULAS.

Integration by parts.

$$\int_\Omega f\partial_i g = -\int_\Omega g\partial_i f + \int_{\partial\Omega} fg\nu_i.$$

First Green's identity.

$$\int_\Omega \langle \nabla f, \nabla g\rangle = -\int_\Omega f\Delta g + \int_{\partial\Omega} f\partial_\nu g.$$

Second Green's identity.

$$\int_\Omega (f\Delta g - g\Delta f) = \int_{\partial\Omega} (f\partial_\nu g - g\partial_\nu f).$$

Polar coordinates.

$$\begin{aligned}\int_{B_R(0)} f &= \int_0^R \left(\int_{\partial B_r(0)} f\, ds\right) dr \\ &= \int_0^R \left(\int_{S^{n-1}} f\, d\omega\right) r^{n-1}\, dr.\end{aligned}$$

Averages.

$$\begin{aligned}⨍_{B_R(0)} f &= \frac{1}{\operatorname{vol}(B_R(0))}\int_{B_R(0)} f \\ &= \frac{1}{\alpha(n)R^n}\int_{B_R(0)} f.\end{aligned}$$

$$\begin{aligned}⨍_{\partial B_R(0)} f &= \frac{1}{\operatorname{vol}(\partial B_R(0))}\int_{\partial B_R(0)} f \\ &= \frac{1}{n\alpha(n)R^{n-1}}\int_{\partial B_R(0)} f.\end{aligned}$$

Fundamental solution. $\Phi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$, $n \geq 3$, given by

$$\Phi(x) = \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}},$$

satisfies

$$-\Delta \Phi = \delta_0.$$

Representation formula. If u solves

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases}$$

then

$$u(x) = \int_\Omega f(y)G(x,y)\,dy - \int_{\partial\Omega} g(y)\partial_\nu G(x,y)\,ds(y),$$

where $G(x,y)$ is the Green function satisfying

$$\begin{cases} -\Delta G = \delta_x, & \text{in } \Omega, \\ G = 0, & \text{on } \partial\Omega. \end{cases}$$

Backward and forward wave solution. $u(t,x) = F(x+ct) + G(x-ct)$ solves

$$u_{tt} - c^2 u_{xx} = 0.$$

D'Alembert's formula.

$$u(t,x) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

solves

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & -\infty < x < \infty,\ t > 0, \\ u(0,x) = f(x),\ u_t(0,x) = g(x), & -\infty < x < \infty. \end{cases}$$

Duhamel's formula. For $s \geq 0$, let $v(t,s,x)$ be the solution, depending on the parameter s, of

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0, & -\infty < x < \infty,\ t > 0, \\ v(0,s,x) = 0,\ v_t(0,s,x) = f(s,x), & -\infty < x < \infty. \end{cases}$$

Then the function

$$u(t,x) = \int_0^t v(t-s,s,x)\,ds$$

solves

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(t,x), & -\infty < x < \infty,\ t > 0, \\ u(0,x) = 0,\ u_t(0,x) = 0, & -\infty < x < \infty. \end{cases}$$

Derivative of a distribution.

$$\langle D^\alpha \varphi, f\rangle = (-1)^{|\alpha|}\langle \varphi, D^\alpha f\rangle,$$

$f \in C_c^\infty(\Omega)$.

Multinomial theorem.

$$(x_1 + \cdots + x_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha,$$

where

$$\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}.$$

Leibniz's formula or product rule in several variables.

$$D^\alpha(fg) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta f D^{\alpha-\beta} g,$$

where $f, g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!},$$

and $\beta \leq \alpha$ means $\beta_i \leq \alpha_i$, $i = 1, \ldots, n$.

Taylor's formula in several variables and in multi-index notation.

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1}),$$

for each $k = 1, 2, \ldots$.