# VANDERBILT UNIVERSITY, MATH 234 SPRING 14: ELEMENTARY NOTIONS OF DISTRIBUTIONS.

## 1. Multi-indices.

A vector of the form $\alpha = (\alpha_1, \dots, \alpha_n)$, where each component $\alpha_i$ is a non-negative integer, is called a *multi-index*. Its *order* is defined as

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Given a multi-index $\alpha$, we define

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f.$$

We also define the factorial of a multi-index as

$$\alpha! = \alpha_1! \cdots \alpha_n!,$$

and multi-index powers of an element $x \in \mathbb{R}^n$ as

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

If $k$ is a non-negative integer, we also set

$$D^k f(x) = \Big\{ D^\alpha f(x) \,\Big|\, |\alpha| = k \Big\}.$$

The sums

$$\sum_{|\alpha|=k} \quad \text{and} \quad \sum_{|\alpha|\leq k}$$

mean, respectively, sum over all multi-indices $\alpha$ such that $|\alpha| = k$ and $|\alpha| \leq k$.

**Problem 1.1.** Let $u : \mathbb{R}^4 \to \mathbb{R}$ be the function

$$u(x_1, x_2, x_3, x_4) = e^{x_1^2 + x_2^2} \big( x_3 x_4 - x_1^3 \sin(x_1 x_3) \big).$$

(a) Find $D^2 u(x)$.
(b) Find

$$\sum_{|\alpha|\leq 3} D^\alpha u(x).$$

**Solution.** The multi-indices with $|\alpha| = 2$ are $(2,0,0,0)$, $(0,2,0,0)$, $(0,0,2,0)$, $(0,0,0,2)$, $(1,1,0,0)$, $(1,0,1,0)$, etc. The corresponding derivatives are easily computed. One also easily computes the derivatives in part (b).

The following formulas are useful when we deal with functions with several variables. Students are encouraged to "play" with them (for instance, choose $|\alpha| = k = 2$ or 3, or even 4, and write explicitly these formulas for functions of two or three variables).

*Multinomial theorem:*

$$(x_1 + \cdots + x_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha,$$

where

$$\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}.$$

*Leibniz's formula or product rule in several variables:*

$$D^\alpha(fg) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta f D^{\alpha-\beta} g,$$

where $f, g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!},$$

and $\beta \le \alpha$ means $\beta_i \le \alpha_i$, $i = 1, \ldots, n$.

*Taylor's formula in several variables and in multi-index notation:*

$$f(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1}),$$

for each $k = 1, 2, \ldots$.

## 2. Test functions.

Let $\Omega \subseteq \mathbb{R}^n$ be a domain in $\mathbb{R}^n$, and let $C_c^\infty(\Omega)$ be the space of $C^\infty$ functions on $\Omega$ with compact support. I.e.,

$$C_c^\infty(\Omega) = \left\{ f \in C^\infty(\Omega) \;\middle|\; \text{supp}(f) \subset \Omega, \text{ and } \text{supp}(f) \text{ is compact} \right\},$$

where $\text{supp}(f)$ denotes the support of $f$. An element of $C_c^\infty(\Omega)$ is called a *test function.*

**Problem 2.1.** Show that $C_c^\infty(\Omega)$ forms a vector space.

**Solution.** If $f, g \in C^\infty(\Omega)$, then

$$\text{supp}(f+g) \subseteq \text{supp}(f) \cup \text{supp}(g),$$

Since $\text{supp}(f+g)$ is closed by the definition of support, and $\text{supp}(f) \cup \text{supp}(g)$ is bounded since both $\text{supp}(f)$ and $\text{supp}(g)$ are, we conclude that $\text{supp}(f+g)$ is a compact set contained in $\Omega$. Thus $f+g$ is a function with support that is compact and contained in $\Omega$. The remaining axioms of vector spaces are trivially verified.

**Definition 2.2.** We say that a sequence of functions $\{f_j\}_{j=1}^\infty \subset C_c^\infty(\Omega)$ converges, when $j \to \infty$, to $f \in C_c^\infty(\Omega)$, in the sense of $C_c^\infty(\Omega)$, if: (1) there exists a compact set $K \subset \Omega$ such that $\text{supp}(f_j) \subset K$ for all $j$, and (2) $D^\alpha f_j$ converges uniformly to $D^\alpha f$ for every multi-index $\alpha$. Sometimes we say simply $f_j$ converges to $f$ in $C_c^\infty(\Omega)$, and write "$f_j \to f$ in $C_c^\infty(\Omega)$."

In the above, uniform convergence means the following. A sequence of functions $\{f_j\}_{j=1}^\infty$, where $f_j : \Omega \to \mathbb{R}$, converges uniformly to a function $f : \Omega \to \mathbb{R}^n$, if, given $\varepsilon > 0$, there exists a $N > 0$, such that

$$|f_j(x) - f(x)| < \varepsilon,$$

for every $j \geq N$ and every $x \in \Omega$.

**Problem 2.3.** Consider the sequences of functions $\{f_j\}_{j=1}^{\infty}$ and $\{g_j\}_{j=1}^{\infty}$, where $f_j : [0,1] \to \mathbb{R}$ is given by $f_j(x) = \frac{x^j}{j!}e^{-jx}$, and $g_j : [0,1] \to \mathbb{R}$ is given by $g_j(x) = x^j$.

(a) Show that, for any fixed non-negative integer $k$, $D^k f_j$ converges uniformly to zero when $j \to \infty$.

(b) Show that, for any fixed $x_0 \in [0,1)$, $g_j(x_0)$ converges to zero when $j \to \infty$, but that $g_j$ does not converge to zero uniformly when $j \to \infty$.

**Solution.** (a) Let $\varepsilon > 0$ be given. Since we are interested in taking the limit $j \to \infty$, we can assume without loss of generality that $j > k$. Using Leibniz's formula (see above),

$$D^k(x^j e^{-jx}) = \sum_{\ell=0}^{k} \binom{k}{\ell} D^\ell x^j D^{k-\ell} e^{-jx} = \sum_{\ell=0}^{k} \frac{k!}{(k-\ell)!\ell!} j(j-1)\cdots(j-\ell)x^{j-\ell}(-j)^{k-\ell}e^{-jx}.$$

Thus

$$\left| D^k \left( \frac{x^j}{j!} e^{-jx} \right) \right| \leq \sum_{\ell=0}^{k} \frac{k!}{(k-\ell)!\ell!} x^{j-\ell} j^{k-\ell} e^{-jx}, \tag{1}$$

where we used that $|(-j)^{k-\ell}| = j^{k-\ell}$ and

$$\frac{j(j-1)\cdots(j-\ell)}{j!} \leq 1.$$

As $e^{-jx} \leq 1$ for $x \in [0,1]$, and $x^{j-\ell} \leq \left(\frac{1}{2}\right)^{j-\ell}$ for $x \in [0, \frac{1}{2}]$, we obtain that

$$\left| D^k \left( \frac{x^j}{j!} e^{-jx} \right) \right| \leq \sum_{\ell=0}^{k} \frac{k!}{(k-\ell)!\ell!} \left(\frac{1}{2}\right)^{j-\ell} j^{k-\ell}, \ \text{ for } x \in [0, \frac{1}{2}]. \tag{2}$$

As an exponential increases faster than any polynomial,

$$\lim_{j\to\infty} \left(\frac{1}{2}\right)^{j-\ell} j^{k-\ell} = \lim_{j\to\infty} 2^{-(j-\ell)} j^{k-\ell} = 0.$$

Therefore, there exists a $N_1 > 0$ such that

$$\sum_{\ell=0}^{k} \frac{k!}{(k-\ell)!\ell!} \left(\frac{1}{2}\right)^{j-\ell} j^{k-\ell} < \varepsilon, \ \text{ for } j \geq N_1, \ \text{ and } x \in [0, \frac{1}{2}]. \tag{3}$$

To treat the case $x \geq \frac{1}{2}$, use the fact that $e^{-jx}$ is decreasing, and that $0 \leq x^{j-\ell} \leq 1$, to obtain from (1),

$$\left| D^k \left( \frac{x^j}{j!} e^{-jx} \right) \right| \leq \sum_{\ell=0}^{k} \frac{k!}{(k-\ell)!\ell!} j^{k-\ell} e^{-\frac{j}{2}}, \ \text{ for } x \in [\frac{1}{2}, 1]. \tag{4}$$

Using again that an exponential increases faster than any polynomial,

$$\lim_{j\to\infty} e^{-\frac{j}{2}} j^{k-\ell} = 0.$$

Therefore, there exists a $N_2 > 0$ such that

$$\sum_{\ell=0}^{k} \frac{k!}{(k-\ell)!\ell!} j^{k-\ell} e^{-\frac{j}{2}} < \varepsilon, \ \text{ for } j \geq N_2, \ \text{ and } x \in [\frac{1}{2}, 1]. \tag{5}$$

Set $N = \max\{N_1, N_2\}$. Then, in light of (2), (3), (4), and (5), we obtain

$$\left| D^k \left( \frac{x^j}{j!} e^{-jx} \right) \right| < \varepsilon, \text{ for } j \geq N, \text{ and any } x \in [0,1],$$

what shows that $D^k f_j$ converges uniformly to zero.

(b) Since $r^j \to 0$ when $j \to \infty$ for any $|r| < 1$, it follows that $x^j \to 0$ for $x \in [0,1)$. But if $x = 1$, $x^j = 1$ for all $j$.

## 3. Distributions.

**Definition 3.1.** Let $\Omega \subseteq \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. A map $\varphi : C_c^\infty(\Omega) \to \mathbb{R}$ is said to be continuous if $\varphi(f_j) \to \varphi(f)$ for every sequence $\{f_j\}_{j=1}^\infty$ such that $f_j \to f$ in $C_c^\infty(\Omega)$.

**Definition 3.2.** A distribution on $C_c^\infty(\Omega)$ is a map $\varphi : C_c^\infty(\Omega) \to \mathbb{R}$ that is continuous and linear. The space of all distributions on $C_c^\infty(\Omega)$ is denoted $\mathcal{D}(\Omega)$.

Distributions are also called *generalized functions.* If $\varphi$ is a distribution and $f$ a test function, it is customary to write $\langle \varphi, f \rangle$ to denote $\varphi(f)$.

The Dirac-delta function is the distribution $\delta$ given by

$$\langle \delta, f \rangle = \delta(f) = f(0),$$

for any test function $f$.

**Problem 3.3.** Show that $\delta$, as defined above, is in fact a distribution, i.e., it is linear and continuous.

**Solution.** Linearity is immediate. For continuity, let $\{f_j\}_{j=1}^\infty$ be a sequence converging in $C_c^\infty(\mathbb{R}^n)$ to a limit $f$. Then in particular $f_j(0) \to f(0)$. Then

$$\lim_{j \to \infty} \langle \delta, f_j \rangle = \lim_{j \to \infty} f_j(0) = f(0) = \langle \delta, f \rangle,$$

showing the result.

Despite its name, the Dirac-delta function is *not* a function, and it does not make sense to talk about its value at one point, e.g., $\delta(x)$. Sometimes one writes

$$\int_{\mathbb{R}^n} f(x) \delta(x)\, dx = f(0), \tag{6}$$

but this is really meant as $\delta(f) = f(0)$, and point-wise values of $\delta$, i.e., $\delta(x)$, are not defined. To see this, suppose it were the case. Then, take the following sequence of functions

$$f_\varepsilon(x) = \begin{cases} \exp\left( -\frac{\varepsilon^2}{\varepsilon^2 - |x|^2} \right), & |x| < \varepsilon, \\ 0, & |x| \geq \varepsilon. \end{cases}$$

It is not difficult to see that $f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$. If $\delta(x)$ were a function, then

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \delta(x) f_\varepsilon(x)\, dx &= \lim_{\varepsilon \to 0} \int_{B_\varepsilon(0)} \delta(x) f_\varepsilon(x)\, dx \\ &\leq e^{-1} \lim_{\varepsilon \to 0} \int_{B_\varepsilon(0)} \delta(x)\, dx \\ &= 0, \end{aligned}$$

since

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon(0)} v(x)\,dx = 0$$

for any integrable function $v$, and (6) would imply that $\delta(x)$ is integrable. However, this contradicts (6) since it should give

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} \delta(x)f_\varepsilon(x)\,dx = \lim_{\varepsilon\to 0} f_\varepsilon(0) = e^{-1}.$$

There is nothing special about the point $x = 0$, and the Dirac-delta function at $x$ is the distribution $\delta_x$ given by

$$\langle \delta_x, f\rangle = \delta_x(f) = f(x),$$

where $f$ is a test function.

Sometimes, one also sees the Dirac-delta "defined" by the following properties

$$\delta(x - x_0) = \begin{cases} 0, & x \neq x_0, \\ \infty, & x = x_0, \end{cases} \quad \text{and} \quad \int_{\mathbb{R}^n} f(x)\delta(x - x_0)\,dx = f(x_0). \tag{7}$$

Again, this formulas are not mathematically precise. They should be understood as follows. Define the function

$$\eta(x) = \begin{cases} N\exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where $N$ is a constant chosen so that

$$\int_{\mathbb{R}^n} \eta = 1.$$

Then, for fixed $x_0 \in \mathbb{R}^n$, define

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x - x_0}{\varepsilon}\right).$$

Notice that $\operatorname{supp}(\eta_\varepsilon) \subseteq B_\varepsilon(x_0)$ and its integral equals one. If $f$ is a test function, then

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x)f(x)\,dx = \int_{B_\varepsilon(x_0)} \eta_\varepsilon(x)f(x)\,dx.$$

Since $f$ is continuous, it has a maximum and a minimum on the closed ball $\overline{B_\varepsilon(x_0)}$, so

$$\min_{B_\varepsilon(x_0)} f(x) = \min_{B_\varepsilon(x_0)} f(x)\int_{B_\varepsilon(x_0)} \eta_\varepsilon(x)\,dx \leq \int_{B_\varepsilon(x_0)} \eta_\varepsilon(x)f(x)\,dx \leq \max_{B_\varepsilon(x_0)} f(x)\int_{B_\varepsilon(x_0)} \eta_\varepsilon(x)f(x)\,dx = \max_{B_\varepsilon(x_0)} f(x).$$

But

$$\lim_{\varepsilon\to 0}\min_{B_\varepsilon(x_0)} f(x) = f(x_0) = \lim_{\varepsilon\to 0}\max_{B_\varepsilon(x_0)} f(x),$$

and therefore

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} \eta_\varepsilon(x)f(x)\,dx = f(x_0) = \delta_{x_0}(f).$$

Now, if in the above limit we forget about the integral and the function $f$, and take the limit $\varepsilon \to 0$ of $\eta_\varepsilon$, the result will be zero everywhere, except at $x_0$, where it blows up; moreover, the integral of $\eta_\varepsilon$ is always equal to one. These are exactly the features described in (7). Thus, heuristically we think of the distribution $\delta_{x_0}$ as the "limit" of the functions $\eta_\varepsilon$.

## 4. Derivatives and weak solutions.

Now we have the tools necessary to make sense of the formula

$$\Delta_y \Gamma(x-y) = -\delta_x \tag{8}$$

saw in class, where $\Gamma$ is the fundamental solution of the Laplacian. For this, let us define what we mean by the derivative of a distribution.

**Definition 4.1.** Let $\varphi \in \mathcal{D}(\Omega)$. Its derivative $D^\alpha \varphi$, where $\alpha$ is a multi-index, is the distribution given by

$$\langle D^\alpha \varphi, f \rangle = (-1)^{|\alpha|} \langle \varphi, D^\alpha f \rangle.$$

The derivative of a distribution is also called a *weak derivative*.

**Problem 4.2.** Show that $D^\alpha \varphi$, as above defined, is in fact a distribution.

**Solution.** Linearity is trivial. For continuity, let $\{f_j\}_{j=1}^\infty \subset C_c^\infty(\Omega)$ be a sequence converging in $C_c^\infty(\Omega)$ to a limit $f$. By definition of $D^\alpha \varphi$,

$$\lim_{j\to\infty} \langle D^\alpha \varphi, f_j \rangle = (-1)^{|\alpha|} \lim_{j\to\infty} \langle \varphi, D^\alpha f_j \rangle.$$

From the definition of convergence in $C_c^\infty(\Omega)$, we have that $D^\alpha f_j$ converges uniformly to $D^\alpha f$ on a compact set containing the supports of the functions $f_j$. Thus $D^\alpha f_j \to D^\alpha f$ in $C_c^\infty(\Omega)$. Since $\varphi$ is continuous, it follows that

$$\lim_{j\to\infty} \langle \varphi, D^\alpha f_j \rangle = \langle \varphi, D^\alpha f \rangle,$$

thus

$$\lim_{j\to\infty} \langle D^\alpha \varphi, f_j \rangle = (-1)^{|\alpha|} \langle \varphi, D^\alpha f \rangle = \langle D^\alpha \varphi, f \rangle.$$

**Problem 4.3.** Let $u : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be the step (or Heaviside) function

$$u(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

(a) Show that $u$ defines a distribution via

$$\langle u, f \rangle = \int_{-\infty}^{\infty} u(x) f(x)\, dx.$$

(b) Show that the weak derivative of $u$ is the Dirac-delta at zero, i.e., $u' = \delta_0$.

**Solution.** (a) Since $f$ has compact support, $\operatorname{supp}(f) \subset [-R, R]$ for some $R$. Then

$$\langle u, f \rangle = \int_{-\infty}^{\infty} u(x) f(x)\, dx, = \int_0^R f(x)\, dx,$$

thus the expression is well-defined. Linearity follows from linearity of the integral. If $f_j \to f$ in $C_c^\infty(\mathbb{R})$, then, by the definition of convergence in $C_c^\infty(\Omega)$, $\operatorname{supp}(f_j) \subset [-R, R]$ for all $j$ and some $R > 0$. Therefore,

$$\lim_{j\to\infty} \langle u, f_j \rangle = \lim_{j\to\infty} \int_0^R f_j(x)\, dx = \int_0^R \lim_{j\to\infty} f_j(x)\, dx = \int_0^R f(x)\, dx = \int_{-\infty}^{\infty} u(x) f(x)\, dx = \langle u, f \rangle.$$

(b) Let $R > 0$ be such that $\operatorname{supp}(f) \subset [-R, R]$, and compute

$$\begin{aligned}\langle u, f' \rangle &= \int_{-\infty}^{\infty} u(x) f'(x)\, dx = \int_0^R f'(x)\, dx = f(x)\Big|_0^R = f(R) - f(0) \\ &= -f(0) = -\langle \delta, f \rangle,\end{aligned}$$

where we used that $f(R) = 0$ since $\operatorname{supp}(f) \subset [-R, R]$. On the other hand, by definition,

$$\langle u', f \rangle = -\langle u, f' \rangle,$$

which shows the result.

Equality (8) can now be understood as follows. The function $\Gamma(x - y)$ defined on $\Omega\backslash\{x\}$, can be viewed as a distribution $\Gamma_x$ if we set

$$\langle \Gamma_x, f \rangle = \int_\Omega \Gamma(x - y) f(y)\, dy,$$

where $f \in C_c^\infty(\Omega)$.

**Problem 4.4.** Show that $\Gamma_x$, as above defined, is in fact a distribution. You are allowed to use the results of previous assignments and what was done in class.

**Solution.** If $u$ is integrable in $\Omega$, then it defines a distribution by

$$\langle u, f \rangle = \int_\Omega u(x) f(x)\, dx,$$

$f \in C_c^\infty(\Omega)$. To see this, first notice that if we let $K \subset \Omega$ be a compact set such that $\operatorname{supp}(f) \subset K$, then

$$\left| \int_\Omega u(x) f(x)\, dx, \right| \leq M \int_K u(x)\, dx < \infty,$$

since $u$ is integrable, and where we used that $|f| \leq M$ for some $M$. Thus, $\langle u, f \rangle$ is well-defined. Linearity follows from linearity of the integral. Finally, if $f_j \to f$ in $C_c^\infty(\Omega)$ and we choose a compact set $K \subset \Omega$ such that $\operatorname{supp}(f_j) \subset K$ for all $j$ (which exists by the definition of convergence in $C_c^\infty(\Omega)$), then

$$\begin{aligned}\lim_{j\to\infty} \langle u, f_j \rangle &= \lim_{j\to\infty} \int_K u(x) f_j(x)\, dx = \int_K u(x) \lim_{j\to\infty} f_j(x)\, dx \\ &= \int_K u(x) f(x)\, dx = \int_\Omega u(x) f(x)\, dx = \langle u, f_j \rangle.\end{aligned}$$

The result now follows from the integrability of $\Gamma_x$ over any compact set.

Formula (8) states that the weak derivative of $\Gamma_x$, as a distribution, equals the distribution $-\delta_x$.

**Problem 4.5.** Show the above statement. You are allowed to use the results of previous assignments and what was done in class.

**Solution.** This follows from the proof of existence for the Poisson equation.

The above examples illustrate how it is possible to meaningfully talk about derivatives of functions that are discontinuous, such as $u$, or that blow-up, such as $\Gamma(x - y)$, provided that we enlarge the concept of derivative to include weak derivatives. This is also similar to situations that we studied in class, where we used D'Alembert's formula for the wave equation with discontinuous initial data.

If we expand the concept of derivative to include weak derivatives, it is natural to expect that we can enlarge the notion of solution of a PDE and talk about *weak solutions*, where the derivatives of the solutions are understood as weak derivatives. Thus, for instance, exercise 4.3 says that we can view $u$ is a weak solution to the PDE

$$u' = \delta_0.$$

The precise definition of weak solutions varies according to the specific equation at hand. But as a general rule, the idea is that weak solutions will always involve considering some derivatives as weak derivatives, or some similar variation.

*URL*: `http://www.disconzi.net/Teaching/MAT234-Spring-14/MAT234-Spring-14.html`