VANDERBILT UNIVERSITY, MATH 234 SPRING 14: ELEMENTARY NOTIONS OF DISTRIBUTIONS.

1. Multi-indices.

A vector of the form $\alpha = (\alpha_1, \dots, \alpha_n)$, where each component α_i is a non-negative integer, is called a *multi-index*. Its *order* is defined as

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Given a multi-index α, we define

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f.$$

We also define the factorial of a multi-index as

$$\alpha! = \alpha_1! \cdots \alpha_n!,$$

and multi-index powers of an element $x \in \mathbb{R}^n$ as

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

If k is a non-negative integer, we also set

$$D^k f(x) = \left\{ D^\alpha f(x) \,\Big|\, |\alpha| = k \right\}.$$

The sums

$$\sum_{|\alpha|=k} \text{ and } \sum_{|\alpha|\leq k}$$

mean, respectively, sum over all multi-indices α such that $|\alpha| = k$ and $|\alpha| \leq k$.

Problem 1.1. Let $u : \mathbb{R}^4 \to \mathbb{R}$ be the function

$$u(x_1, x_2, x_3, x_4) = e^{x_1^2 + x_2^2}\big(x_3 x_4 - x_1^3 \sin(x_1 x_3)\big).$$

(a) Find $D^2 u(x)$.
(b) Find

$$\sum_{|\alpha|\leq 3} D^\alpha u(x).$$

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The following formulas are useful when we deal with functions with several variables. Students are encouraged to "play" with them (for instance, choose $|\alpha| = k = 2$ or 3, or even 4, and write explicitly these formulas for functions of two or three variables).

Multinomial theorem:

$$(x_1 + \cdots + x_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha,$$

where

$$\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}.$$

Leibniz's formula or product rule in several variables:

$$D^\alpha(fg) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta f D^{\alpha-\beta} g,$$

where $f, g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!},$$

and $\beta \leq \alpha$ means $\beta_i \leq \alpha_i$, $i = 1, \ldots, n$.

Taylor's formula in several variables and in multi-index notation:

$$f(x) = \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1}),$$

for each $k = 1, 2, \ldots$.

2. Test functions.

Let $\Omega \subseteq \mathbb{R}^n$ be a domain in $\mathbb{R}^n$, and let $C_c^\infty(\Omega)$ be the space of C^∞ functions on Ω with compact support. I.e.,

$$C_c^\infty(\Omega) = \left\{ f \in C^\infty(\Omega) \;\middle|\; \text{supp}(f) \subset \Omega, \text{ and } \text{supp}(f) \text{ is compact } \right\},$$

where $\text{supp}(f)$ denotes the support of f. An element of $C_c^\infty(\Omega)$ is called a *test function.*

Problem 2.1. Show that $C_c^\infty(\Omega)$ forms a vector space.

Definition 2.2. We say that a sequence of functions $\{f_j\}_{j=1}^\infty \subset C_c^\infty(\Omega)$ converges, when $j \to \infty$, to $f \in C_c^\infty(\Omega)$, in the sense of $C_c^\infty(\Omega)$, if: (1) there exists a compact set $K \subset \Omega$ such that $\text{supp}(f_j) \subset K$ for all j, and (2) $D^\alpha f_j$ converges uniformly to $D^\alpha f$ for every multi-index α. Sometimes we say simply f_j converges to f in $C_c^\infty(\Omega)$, and write "$f_j \to f$ in $C_c^\infty(\Omega)$."

In the above, uniform convergence means the following. A sequence of functions $\{f_j\}_{j=1}^\infty$, where $f_j : \Omega \to \mathbb{R}$, converges uniformly to a function $f : \Omega \to \mathbb{R}^n$, if, given $\varepsilon > 0$, there exists a $N > 0$, such that

$$|f_j(x) - f(x)| < \varepsilon,$$

for every $j \geq N$ and every $x \in \Omega$.

Problem 2.3. Consider the sequences of functions $\{f_j\}_{j=1}^\infty$ and $\{g_j\}_{j=1}^\infty$, where $f_j : [0,1] \to \mathbb{R}$ is given by $f_j(x) = \frac{x^j}{j!} e^{-jx}$, and $g_j : [0,1] \to \mathbb{R}$ is given by $g_j(x) = x^j$.

(a) Show that, for any fixed non-negative integer k, $D^k f_j$ converges uniformly to zero when $j \to \infty$.

(b) Show that, for any fixed $x_0 \in [0,1)$, $g_j(x_0)$ converges to zero when $j \to \infty$, but that g_j does not converge to zero uniformly when $j \to \infty$.

3. Distributions.

Definition 3.1. Let $\Omega \subseteq \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. A map $\varphi : C_c^\infty(\Omega) \to \mathbb{R}$ is said to be continuous if $\varphi(f_j) \to \varphi(f)$ for every sequence $\{f_j\}_{j=1}^\infty$ such that $f_j \to f$ in $C_c^\infty(\Omega)$.

Definition 3.2. A distribution on $C_c^\infty(\Omega)$ is a map $\varphi : C_c^\infty(\Omega) \to \mathbb{R}$ that is continuous and linear. The space of all distributions on $C_c^\infty(\Omega)$ is denoted $\mathcal{D}(\Omega)$.

Distributions are also called *generalized functions*. If φ is a distribution and f a test function, it is customary to write $\langle \varphi, f \rangle$ to denote $\varphi(f)$.

The Dirac-delta function is the distribution δ given by

$$\langle \delta, f \rangle = \delta(f) = f(0),$$

for any test function f.

Problem 3.3. Show that δ, as defined above, is in fact a distribution, i.e., it is linear and continuous.

Despite its name, the Dirac-delta function is *not* a function, and it does not make sense to talk about its value at one point, e.g., $\delta(x)$. Sometimes one writes

$$\int_{\mathbb{R}^n} f(x)\delta(x)\,dx = f(0), \tag{1}$$

but this is really meant as $\delta(f) = f(0)$, and point-wise values of δ, i.e., $\delta(x)$, are not defined. To see this, suppose it were the case. Then, take the following sequence of functions

$$f_\varepsilon(x) = \begin{cases} \exp\left(-\frac{\varepsilon^2}{\varepsilon^2 - |x|^2}\right), & |x| < \varepsilon, \\ 0, & |x| \geq \varepsilon. \end{cases}$$

It is not difficult to see that $f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$. If $\delta(x)$ were a function, then

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \delta(x) f_\varepsilon(x)\,dx &= \lim_{\varepsilon \to 0} \int_{B_\varepsilon(0)} \delta(x) f_\varepsilon(x)\,dx \\ &\leq e^{-1} \lim_{\varepsilon \to 0} \int_{B_\varepsilon(0)} \delta(x)\,dx \\ &= 0, \end{aligned}$$

since

$$\lim_{\varepsilon \to 0} \int_{B_\varepsilon(0)} v(x)\,dx = 0$$

for any integrable function v, and (1) would imply that $\delta(x)$ is integrable. However, this contradicts (1) since it should give

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \delta(x) f_\varepsilon(x)\,dx = \lim_{\varepsilon \to 0} f_\varepsilon(0) = e^{-1}.$$

There is nothing special about the point $x = 0$, and the Dirac-delta function at x is the distribution δ_x given by

$$\langle \delta_x, f \rangle = \delta_x(f) = f(x),$$

where f is a test function.

Sometimes, one also sees the Dirac-delta "defined" by the following properties

$$\delta(x - x_0) = \begin{cases} 0, & x \neq x_0, \\ \infty, & x = x_0, \end{cases} \quad \text{and} \quad \int_{\mathbb{R}^n} f(x)\delta(x - x_0)\,dx = f(x_0). \tag{2}$$

Again, this formulas are not mathematically precise. They should be understood as follows. Define the function

$$\eta(x) = \begin{cases} N \exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where N is a constant chosen so that

$$\int_{\mathbb{R}^n} \eta = 1.$$

Then, for fixed $x_0 \in \mathbb{R}^n$, define

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x - x_0}{\varepsilon}\right).$$

Notice that $\operatorname{supp}(\eta_\varepsilon) \subseteq B_\varepsilon(x_0)$ and its integral equals one. If f is a test function, then

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x) f(x)\, dx = \int_{B_\varepsilon(x_0)} \eta_\varepsilon(x) f(x)\, dx.$$

Since f is continuous, it has a maximum and a minimum on the closed ball $\overline{B_\varepsilon(x_0)}$, so

$$\min_{\overline{B_\varepsilon(x_0)}} f(x) = \min_{\overline{B_\varepsilon(x_0)}} f(x) \int_{B_\varepsilon(x_0)} \eta_\varepsilon(x)\, dx \leq \int_{B_\varepsilon(x_0)} \eta_\varepsilon(x) f(x)\, dx \leq \max_{\overline{B_\varepsilon(x_0)}} f(x) \int_{B_\varepsilon(x_0)} \eta_\varepsilon(x) f(x)\, dx = \max_{\overline{B_\varepsilon(x_0)}} f(x).$$

But

$$\lim_{\varepsilon\to 0} \min_{\overline{B_\varepsilon(x_0)}} f(x) = f(x_0) = \lim_{\varepsilon\to 0} \max_{\overline{B_\varepsilon(x_0)}} f(x),$$

and therefore

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \eta_\varepsilon(x) f(x)\, dx = f(x_0) = \delta_{x_0}(f).$$

Now, if in the above limit we forget about the integral and the function f, and take the limit $\varepsilon \to 0$ of η_ε, the result will be zero everywhere, except at x_0, where it blows up; moreover, the integral of η_ε is always equal to one. These are exactly the features described in (2). Thus, heuristically we think of the distribution δ_{x_0} as the "limit" of the functions η_ε.

4. Derivatives and weak solutions.

Now we have the tools necessary to make sense of the formula

$$\Delta_y \Gamma(x - y) = -\delta_x \tag{3}$$

saw in class, where Γ is the fundamental solution of the Laplacian. For this, let us define what we mean by the derivative of a distribution.

Definition 4.1. Let $\varphi \in \mathcal{D}(\Omega)$. Its derivative $D^\alpha\varphi$, where α is a multi-index, is the distribution given by

$$\langle D^\alpha \varphi, f\rangle = (-1)^{|\alpha|}\langle \varphi, D^\alpha f\rangle.$$

The derivative of a distribution is also called a *weak derivative*.

Problem 4.2. Show that $D^\alpha\varphi$, as above defined, is in fact a distribution.

Problem 4.3. Let $u : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be the step (or Heaviside) function

$$u(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

(a) Show that u defines a distribution via

$$\langle u, f \rangle = \int_{-\infty}^{\infty} u(x) f(x)\, dx.$$

(b) Show that the weak derivative of u is the Dirac-delta at zero, i.e., $u' = \delta_0$.

Equality (3) can now be understood as follows. The function $\Gamma(x-y)$ defined on $\Omega\backslash\{x\}$, can be viewed as a distribution Γ_x if we set

$$\langle \Gamma_x, f \rangle = \int_{\Omega} \Gamma(x-y) f(y)\, dy,$$

where $f \in C_c^\infty(\Omega)$.

Problem 4.4. Show that Γ_x, as above defined, is in fact a distribution. You are allowed to use the results of previous assignments and what was done in class.

Formula (3) states that the weak derivative of Γ_x, as a distribution, equals the distribution $-\delta_x$.

Problem 4.5. Show the above statement. You are allowed to use the results of previous assignments and what was done in class.

The above examples illustrate how it is possible to meaningfully talk about derivatives of functions that are discontinuous, such as u, or that blow-up, such as $\Gamma(x-y)$, provided that we enlarge the concept of derivative to include weak derivatives. This is also similar to situations that we studied in class, where we used D'Alembert's formula for the wave equation with discontinuous initial data.

If we expand the concept of derivative to include weak derivatives, it is natural to expect that we can enlarge the notion of solution of a PDE and talk about *weak solutions*, where the derivatives of the solutions are understood as weak derivatives. Thus, for instance, exercise 4.3 says that we can view u is a weak solution to the PDE

$$u' = \delta_0.$$

The precise definition of weak solutions varies according to the specific equation at hand. But as a general rule, the idea is that weak solutions will always involve considering some derivatives as weak derivatives, or some similar variation.

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