VANDERBILT UNIVERSITY
MATH 234 — INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS
SOLUTIONS TO SOME PROBLEMS OF THE PRACTICE TEST.

Question 1. Suppose $f : [0,1] \to \mathbb{R}$ can be written as the following series

$$f(x) = \sum_{n=0}^{\infty} \Big(c_n \sin(n\pi x) + d_n \cos(n\pi x) \Big).$$

Show that the coefficients c_n and d_n are given by

$$c_n = 2 \int_0^1 f(x) \sin(n\pi x)\, dx, n = 0, 1, 2, \dots,$$
$$d_n = 2 \int_0^1 f(x) \cos(n\pi x)\, dx,\ n = 1, 2, \dots,$$
$$d_0 = \int_0^1 f(x) \cos(n\pi x)\, dx.$$

Solution. This is similar to what was done in class. Multiply the expression for f by $\sin(m\pi x)$, integrate and use that

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\, dx = \int_0^1 \sin(m\pi x)\cos(n\pi x)\, dx = 0$$

if $m \neq n$, and similarly when we multiply f by $\cos(m\pi x)$.

Question 2. Explain the concept of *well-posedness* for a PDE.

Solution. A initial-boundary value problem is well-posed when (i) there exists a solution (existence); (ii) there is no more than one solution (uniqueness); (iii) small changes in the data yield small changes in the solution (continuity on data).

Question 3. Let $\mathbb{R}^2_+$ be the upper half plane in $\mathbb{R}^2$, i.e.,

$$\mathbb{R}^2_+ = \Big\{ (x, y) \in \mathbb{R}^2 \,\Big|\, y > 0 \Big\}.$$

Consider the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ u = 0 & \text{for } y = 0 \\ \frac{\partial u}{\partial y} = \frac{1}{n}\sin(nx) & \text{for } y = 0, \end{cases} \tag{1}$$

where n is a given positive integer. Use separation of variables to show that the function

$$u(x, y) = \frac{1}{n^2}\frac{e^{ny} - e^{-ny}}{2}\sin(nx) \tag{2}$$

is a solution of (1).

Solution. This follows by direct application of separation of variables.

Question 4. Taking the limit $n \to \infty$ in (1) and (2), what can you conclude about the well-posedness of the boundary value problem (1)?

Solution. When $n \to \infty$ the problem becomes

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ u = 0 & \text{for } y = 0 \\ \frac{\partial u}{\partial y} = 0 & \text{for } y = 0, \end{cases}$$

which has $u = 0$ as solution, while solution (2) diverges to infinity. Hence solutions do not not depend continuously on the data, therefore the problem is not well-posed.

Question 4. Find the expression for the Laplacian in polar coordinates in two-dimensions.

Solution. Check any textbook of multivariable calculus.

Question 5. Consider the equation

$$u_{xx} + 2u_{xy} + u_{yy} = 0.$$

Rewrite the equation in terms of the coordinates $s = x$, $t = x - y$.

Solution. This is a problem from chapter 1 of the textbook (see answers on the back), and it is a straightforward application of the chain rule.

Question 6. Let Ω be a domain in $\mathbb{R}^n$. Show that the Laplacian Δ is a linear map between $C^k(\Omega)$ and $C^{k-2}(\Omega)$, $k \geq 2$.

Solution. This was one of the problems in the extra-credit assignment.

For questions 7 and 8 below, you are allowed to use short-cuts based on your previous experience with these equations. For example, when you separate variables and have to analyze the different cases $\mu < 0$, $\mu = 0$, and $\mu > 0$, you already know that one of these cases will give a solution identically zero — you can simply state that, without showing all the work. You can also use the formulas of problem 1, without redoing all the work.

Question 7. Solve the following boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1,\, t > 0, \\ u(t,0) = 0,\, u(t,1) = 0, & t \geq 0, \\ u(0,x) = f(x),\, u_t(0,x) = g(x), & 0 \leq x \leq 1, \end{cases}$$

where f and g are given functions.

Solution. Similar to what was done in class.

Question 8. Solve the following boundary value problem

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < 1,\, t > 0, \\ u_x(t,0) = 0,\, u_x(t,1) = 0, & t \geq 0, \\ u(0,x) = f(x) & 0 \leq x \leq 1, \end{cases}$$

where f is a given function.

Solution. Similar to what was done in class.

Question 9. Let Ω be a domain in $\mathbb{R}^3$ containing the origin. Show that the function

$$\Gamma(x) = \frac{1}{|x|}$$

satisfies

$$\Delta\Gamma = 0 \text{ in } \Omega\backslash\{0\}.$$

Solution. In spherical coordinates, $\Gamma(r,\theta,\phi) = \Gamma(r) = \frac{1}{r}$. Writing the Laplacian in spherical coordinates (check any multivariable calculus textbook), it is straightforward to check that $\Delta\Gamma = 0$ holds for $r \neq 0$.

Question 10. Let

$$\Omega = \Big\{(x_1, x_2) \in \mathbb{R}^2 \,\Big|\, x_1^2 + x_2^2 < 1\Big\}.$$

Suppose u is a function on Ω that satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 1 & \text{on } \partial\Omega. \end{cases}$$

What can you say about $u(0,0)$?

Solution. Using the mean value formula:

$$u(0,0) = \frac{1}{2\pi}\int_{\partial\Omega} u\, ds = \frac{1}{2\pi}\int_{\partial\Omega} 1\, ds = \frac{1}{2\pi}2\pi = 1.$$

Remark: Strictly speaking, the mean value formula holds for balls $B_r(x) \subset \Omega$ such that $\partial B_r(x) \cap \partial\Omega = \varnothing$. But we can consider the ball $B_{1-\varepsilon}(0)$, $\varepsilon > 0$, use the mean value formula, and take the limit $\varepsilon \to 0$.

In questions 11 to 15 below, Ω is a bounded domain in $\mathbb{R}^n$.

Question 11. Prove uniqueness of solutions to the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where f and g are given functions.

Solution. This was done in class.

Question 12. Prove that the following Neumann problem

$$\begin{cases} \Delta u = 1 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

has no solution (*hint:* integration by parts). What can you say about the solvability of

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases}$$

where f and g are given functions?

Solution. If u is a solution of the second problem above, then integrating the equation $\Delta u = f$ over Ω and integrating by parts yields

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\Omega} f.$$

Using the boundary condition gives

$$\int_{\partial\Omega} g = \int_{\Omega} f.$$

and hence if f and g do not satisfy this relation, the problem has no solution. This is the case, in particular, then $g = 0$ and $f = 1$.

Question 13. Show uniqueness of solutions to

$$\begin{cases} u_{tt} - c^2 \Delta u = F(t,x), & x \in \Omega,\ t > 0, \\ u(t,x) = h(t,x), & x \in \partial\Omega,\ t \geq 0, \\ u(0,x) = f(x),\ u_t(0,x) = g(x), & x \in \Omega, \end{cases}$$

where F, h, f, and g are given functions.

Solution. This was done in class in one spatial dimension, and it was a homework problem in higher dimensions.

Question 14. Let u be a solution of

$$\begin{cases} \Delta u - u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where f is a given function.
Denote by α the maximum value of $|f|$ (the absolute value of f) on $\overline{\Omega}$, i.e.,

$$\alpha = \max_{\overline{\Omega}} |f|$$

Notice that α is a constant.
(a) Define $v = u - \alpha$. Show that v satisfies

$$\begin{cases} \Delta v - v = \alpha + f & \text{in } \Omega, \\ v = -\alpha & \text{on } \partial\Omega. \end{cases}$$

(b) By analyzing both the cases where the maximum of v is in the interior of Ω or on its boundary, show that $v(x) \leq 0$ for any $x \in \Omega$. Conclude that

$$u \leq \alpha.$$

(c) Using similar ideas as above, show that u also satisfies

$$-\alpha \leq u,$$

and therefore any solution u of (3) satisfies

$$|u| \leq \max_{\overline{\Omega}} |f|.$$

Solution. Computing Δv and using that α is constant, we have $\Delta v = \Delta(u - \alpha) = \Delta u$. But $\Delta u = u + f = v + \alpha + f$. Since $v = -\alpha$ on the boundary (because u vanishes on the boundary), this shows (a).

Let x_0 be a maximum of v. If x_0 is an interior point, then $\Delta v(x_0) \leq 0$, so

$$-v(x_0) \geq \Delta v(x_0) - v(x_0) = \alpha + f(x_0).$$

Since $|f| \geq 0$, and α is the maximum of $|f|$, we see that $\alpha + f \geq 0$ always holds, therefore the above gives $-v(x_0) \geq 0$, or $v(x_0) \leq 0$. Since this is the maximum of v, we conclude that $v \leq 0$. If now $x_0 \in \partial\Omega$, then $v(x_0) = -\alpha \leq 0$, and again we conclude that v is non-positive. Hence $u - \alpha \leq 0$, or $u \leq \alpha$. This shows (b)

Set now $w = u + \alpha$. Then $\Delta w = \Delta u = u + f = w - \alpha + f$. Since u vanishes on the boundary, we see that w satisfies

$$\begin{cases} \Delta w - w = f - \alpha & \text{in } \Omega, \\ w = \alpha & \text{on } \partial\Omega. \end{cases}$$

Let x_1 be a minimum of w. If x_1 is an interior point, then $\Delta w(x_1) \geq 0$ and

$$-w(x_1) \leq \Delta w(x_1) - w(x_1) = f(x_1) - \alpha.$$

Since α is the maximum of $|f|$, we see that $f - \alpha \leq 0$ always holds, therefore $-w(x_1) \leq 0$, or $w(x_1) \geq 0$. Since x_1 is the minimum of w, we have $w \geq 0$. If x_1 is on the boundary, then $w(x_1) = \alpha \geq 0$, and we conclude $u \geq -\alpha$.

We obtain that $-\alpha \leq u \leq \alpha$, what shows (c).

Question 15. Let u_1 and u_2 be respectively solutions of

$$\begin{cases} \Delta u - u = f_1 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} \Delta u - u = f_2 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where f_1, f_2 and g are given functions. Using the results of problem 14, what can you say about $u_1 - u_2$?

Solution. Set $v = u_1 - u_2$. Then

$$\begin{cases} \Delta v - v = f_1 - f_2 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

By the previous problem, we have

$$|v| = |u_1 - u_2| \leq \max_{\overline{\Omega}} |f_1 - f_2|,$$

and therefore the solutions u_1 and u_2 are close to each other if the given functions f_1 and f_2 are close to each other. In particular, if $f_1 = f_2$, then $u_1 = u_2$.