VANDERBILT UNIVERSITY
MATH 234 — INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS
PRACTICE TEST.

Question 1. Suppose $f : [0,1] \to \mathbb{R}$ can be written as the following series

$$f(x) = \sum_{n=0}^{\infty} \Big(c_n \sin(n\pi x) + d_n \cos(n\pi x) \Big).$$

Show that the coefficients c_n and d_n are given by

$$c_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx, n = 0,1,2,\dots,$$
$$d_n = 2\int_0^1 f(x)\cos(n\pi x)\,dx,\ n = 1,2,\dots,$$
$$d_0 = \int_0^1 f(x)\cos(n\pi x)\,dx.$$

Question 2. Explain the concept of *well-posedness* for a PDE.

Question 3. Let $\mathbb{R}^2_+$ be the upper half plane in $\mathbb{R}^2$, i.e.,

$$\mathbb{R}^2_+ = \Big\{ (x,y) \in \mathbb{R}^2 \,\Big|\, y > 0 \Big\}.$$

Consider the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ u = 0 & \text{for } y = 0 \\ \frac{\partial u}{\partial y} = \frac{1}{n}\sin(nx) & \text{for } y = 0, \end{cases} \tag{1}$$

where n is a given positive integer. Use separation of variables to show that the function

$$u(x,y) = \frac{1}{n^2}\frac{e^{ny} - e^{-ny}}{2}\sin(nx) \tag{2}$$

is a solution of (1).

Question 4. Taking the limit $n \to \infty$ in (1) and (2), what can you conclude about the well-posedness of the boundary value problem (1)?

Question 4. Find the expression for the Laplacian in polar coordinates in two-dimensions.

Question 5. Consider the equation

$$u_{xx} + 2u_{xy} + u_{yy} = 0.$$

Rewrite the equation in terms of the coordinates $s = x$, $t = x - y$.

Question 6. Let Ω be a domain in $\mathbb{R}^n$. Show that the Laplacian Δ is a linear map between $C^k(\Omega)$ and $C^{k-2}(\Omega)$, $k \geq 2$.

For questions 7 and 8 below, you are allowed to use short-cuts based on your previous experience with these equations. For example, when you separate variables and have to analyze the different *cases $\mu < 0$, $\mu = 0$, and $\mu > 0$, you already know that one of these cases will give a solution identically zero — you can simply state that, without showing all the work. You can also use the formulas of problem 1, without redoing all the work.*

Question 7. Solve the following boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1,\, t > 0, \\ u(t,0) = 0,\, u(t,1) = 0, & t \geq 0, \\ u(0,x) = f(x),\, u_t(0,x) = g(x), & 0 \leq x \leq 1, \end{cases}$$

where f and g are given functions.

Question 8. Solve the following boundary value problem

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < 1,\, t > 0, \\ u_x(t,0) = 0,\, u_x(t,1) = 0, & t \geq 0, \\ u(0,x) = f(x) & 0 \leq x \leq 1, \end{cases}$$

where f is a given function.

Question 9. Let Ω be a domain in $\mathbb{R}^3$ containing the origin. Show that the function

$$\Gamma(x) = \frac{1}{|x|}$$

satisfies

$$\Delta\Gamma = 0 \text{ in } \Omega\backslash\{0\}.$$

Question 10. Let

$$\Omega = \left\{(x_1, x_2) \in \mathbb{R}^2 \,\middle|\, x_1^2 + x_2^2 < 1\right\}.$$

Suppose u is a function on Ω that satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 1 & \text{on } \partial\Omega. \end{cases}$$

What can you say about $u(0,0)$?

In questions 11 to 15 below, Ω is a bounded domain in $\mathbb{R}^n$.

Question 11. Prove uniqueness of solutions to the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where f and g are given functions.

Question 12. Prove that the following Neumann problem

$$\begin{cases} \Delta u = 1 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

has no solution (*hint:* integration by parts). What can you say about the solvability of

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases}$$

where f and g are given functions?

Question 13. Show uniqueness of solutions to

$$\begin{cases} u_{tt} - c^2 \Delta u = F(t,x), & x \in \Omega,\ t > 0, \\ u(t,x) = h(t,x), & x \in \partial\Omega,\ t \geq 0, \\ u(0,x) = f(x),\ u_t(0,x) = g(x), & x \in \Omega, \end{cases}$$

where F, h, f, and g are given functions.

Question 14. Let u be a solution of

$$\begin{cases} \Delta u - u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where f is a given function.
Denote by α the maximum value of $|f|$ (the absolute value of f) on $\overline{\Omega}$, i.e.,

$$\alpha = \max_{\overline{\Omega}} |f|$$

Notice that α is a constant.
(a) Define $v = u - \alpha$. Show that v satisfies

$$\begin{cases} \Delta v - v = \alpha + f & \text{in } \Omega, \\ v = -\alpha & \text{on } \partial\Omega. \end{cases}$$

(b) By analyzing both the cases where the maximum of v is in the interior of Ω or on its boundary, show that $v(x) \leq 0$ for any $x \in \Omega$. Conclude that

$$u \leq \alpha.$$

(c) Using similar ideas as above, show that u also satisfies

$$-\alpha \leq u,$$

and therefore any solution u of (3) satisfies

$$|u| \leq \max_{\overline{\Omega}} |f|.$$

Question 15. Let u_1 and u_2 be respectively solutions of

$$\begin{cases} \Delta u - u = f_1 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} \Delta u - u = f_2 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where f_1, f_2 and g are given functions. Using the results of problem 14, what can you say about $u_1 - u_2$?