By Parker Bordeaux

# Curvature

Today I'm going to talk about how curvature – in particular Gaussian Curvature Named after German mathematiccan

Johann Carl Friedrich – Gauss

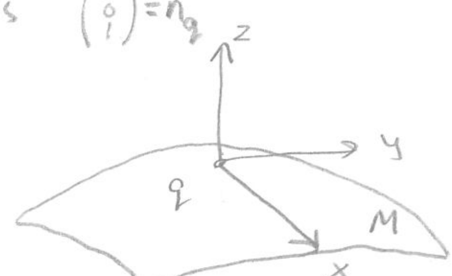
a crude drawing of Carl

plays a role in PDEs.

In actuality today will be a talk about how PDEs play a role in geometry.

My goal is now to explain the notion of curvature – minimal, maximal, and Gaussian – in the easiest way possible i.e. without the notion of covariant derivative or the second fundamental form. If this interests you, look up first fundamental form, second fundamental form (shape operator), connections, or just differential Geometry.

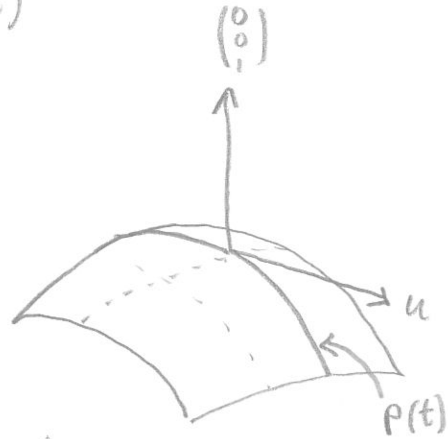
Let $M \subset \mathbb{R}^3$ be a surface s.t. $M$ is the graph of a $C^\infty$ (smooth) function $z = f(x,y)$. For simplication let $M$ pass through the origin, $q$, and its tangent plane at $q$, $T_qM$, is $z=0$ then the unit normal at $q$ is $\begin{pmatrix}0\\0\\1\end{pmatrix} = n_q$



Let $u$ be a unit vector in $T_qM$, $u = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$

let $p$ be the parameterized curve given by slicing $M$ through the plane spanned by $u$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = n_q$,

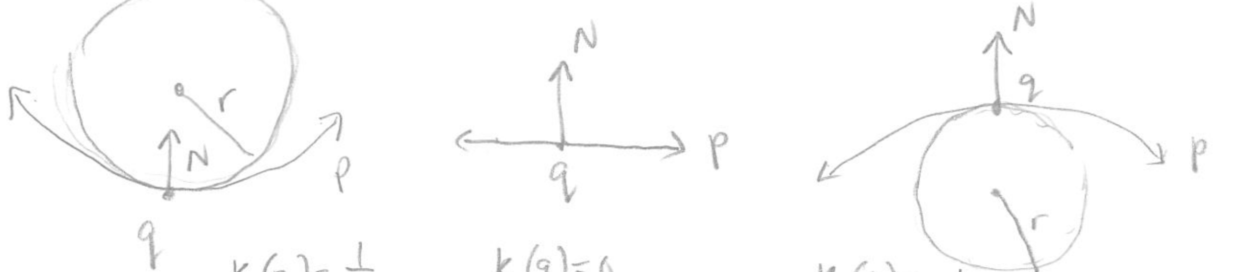
$$p(t) = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} f(u_1 t, u_2 t)$$




$p$ has a signed curvature $k_s$ at $q$ with respect to $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $k_s$ is basically the reciprocal of the radius of the osculating circle to $p$ at $q$

(an osculating circle at a point $q$ to a curve is the circle passing through $q$ and a pair of points $q_\varepsilon$ $q_\delta$ infinitesimally close to $q$. Its center lies on the inner normal line and its curvature is the same as the curve at $q$)

taken with sign as follows:




$k_u(q) = \frac{1}{r}$ $\qquad$ $k_u(q) = 0$ $\qquad$ $k_u(q) = -\frac{1}{r}$

or we see $p''(s)\big|_{s=0} = k_u N(q)$ where $s(t)$ is arc length

and $s(t) = \int_0^t |p'(t)| dt$.

example: circle radius $r$,

then $c''(s)\big|_{s=0} = \frac{-r^2}{r^3} = \frac{-1}{r}$ (this is with respect to the outward normal)

(its normally $\frac{1}{r}$ i.e. inward normal) then $k_u = f_{xx}(0,0)u_1^2 + 2f_{xy}(0,0)u_1u_2 + f_{yy}(0,0)u_2^2$

$\therefore$

$$k_u = [u_1, u_2]\begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{xy}(0,0) & f_{yy}(0,0)\end{bmatrix}\begin{bmatrix} u_1 \\ u_2\end{bmatrix}$$

The principle curvatures, $k_1$ $k_2$, are the largest and smallest possible values, $k_1$ $k_2$, of $k_u$ as $u$ varies over the possible unit tangent vectors, and the unit tangent vectors are called principle directions. (unfortunately these formulas only work at $q$ because derivation relies on $f_x(0) = f_y(0) = 0$)

$K = k_1 k_2$ is the Gaussian Curvature

$H = k_1 + k_2$ is mean curvature

recall: $\mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0$ the minimal surface equation, all minimal surfaces have zero twice mean curvature

$\therefore$ $2H = \mathrm{div}\left(\frac{\nabla f}{1+|\nabla f|^2}\right) \Rightarrow 2k_1 + 2k_2 = \partial_x\left(\frac{f_x}{\sqrt{1+|\nabla f|^2}}\right) + \partial_y\left(\frac{f_y}{\sqrt{1+|\nabla f|^2}}\right)$

But it is easier to find $K$ using the fact that

$K = \det(dN_f)$ where $N_f$ is the Gauss map of $f$. The Gauss map of $f$ is the map that assigns to each point on the surface $M$ the unique unit vector parallel to the unit normal at that point, so

$N_f : M \to S^2 \Rightarrow dN_f : T_qM \to T_{N(q)}S^2$ but $T_qM$ is parallel to $T_{N(q)}S^2$ $\therefore$ $dN_f : T_q(M) \to T_q(M)$

then the linear map $dN_f$ at $q$ $dN_f : T_qM \to T_qM$

works : $\forall\ p(t) \subseteq M$ $p(0) = q$ consider $N \circ p(t) = N(t) \subseteq S^2$

this is restricting the normal vector to the curve $p(t)$. Then $N'(0) = dN_f\,(p'(0))$ the tangent vector (at $q$), is a vector in the tangent plane at $q$ to $M$.

$dN_f$ (at $q$) then measures the rate of change of the normal vector restricted to the curve $p(t)$ at $t=0$ $\therefore$ $dN_f$ (at $q$) measures how the normal vector pulls away from $N(q)$ in a neighborhood of $q$ (ball about $q$).

In coordinates $(x,y) \to (x,y,f(x,y))$

the Gauss map of $f$ $N_f$ is given by

$$N_f = \left(\frac{-\nabla f}{\sqrt{1+|\nabla f|^2}}, \frac{1}{\sqrt{1+|\nabla f|^2}}\right)$$

then $dN_f$

$$dN_f = \begin{pmatrix} \partial_x\left(\frac{-\nabla f}{\sqrt{1+|\nabla f|^2}}\right) & \partial_y\left(\frac{-\nabla f}{\sqrt{1+|\nabla f|^2}}\right) \\ \partial_x\left(\frac{1}{\sqrt{1+|\nabla f|^2}}\right) & \partial_y\left(\frac{1}{\sqrt{1+|\nabla f|^2}}\right) \end{pmatrix}$$

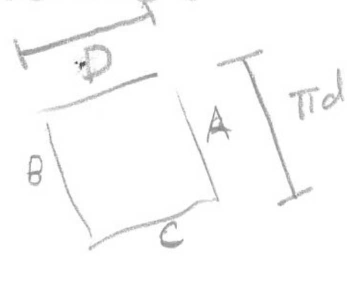
then (*) $$K = \partial_x\left(\frac{-\nabla f}{\sqrt{1+|\nabla f|^2}}\right)\partial_y\left(\frac{1}{\sqrt{1+|\nabla f|^2}}\right) - \partial_x\left(\frac{1}{\sqrt{1+|\nabla f|^2}}\right)\partial_y\left(\frac{-\nabla f}{\sqrt{1+|\nabla f|^2}}\right)$$

(**) $$= \frac{f_{xx}f_{yy} - f_{xy}^2}{(1+|\nabla f|^2)^2} = \frac{\det D^2 f}{(1+|\nabla f|^2)^{(2+2)/2}}$$

(step from (*) $\to$ (**) took up too much room / time to include just a lot of tedious differentials)

Now that we have a slight idea of the basic math of curvature, here is an intuitive explanation of Gaussian curvature:

think of a sheet of paper




it has 0 gaussian curvature, take that sheet and assign equivalence class $A \sim B$, you get a tube


or $B \sim C$ you get a cone


and the gaussian curvature is still 0. Anything that you do to the paper that allows you to flatten back to the sheet without wrinkles or tears preserves its Gaussian curvature (this is because K is a topological invariant) but notice if you try to wrap the sheet around a sphere diameter=d you have to wrinkle the sheet, especially at the edges, to make it fit the sphere's curve. This is because the sphere has positive K and the circumference of a circle drawn on a sphere is at most $\pi d$, the folds are where you have extra circumference. If you try to conform the sheet to a saddle you see you must tear the sheet in the middle to make it lie flat. That is because a surface with negative curvature the circumference of a circle is longer than $\pi d$ $\therefore$ you have to increase the sheet's length in the middle by tearing it. In nature where math principles are personified one sees plants produce curved or wrinkled leaves by altering the rate at which the edged of a leaf grow compared to the center which alters $K$

## A few PDEs related to $K$:

**I: Prescribed Gaussian Curvature:** (open question in geometry)

version 1 : Given a 2-d Riemannian manifold, can we embed it isometrically (even just locally) in $\mathbb{R}^3$?

version 2 : Given a $C^\infty$ function $K(x,y)$, find (even locally) a function $f(x,y)$ which solves

$$\frac{f_{xx}f_{yy} - f_{xy}^2}{(1+|\nabla f|^2)} = K(x,y)$$

Knowing the curvature of $M : z = f(x,y)$ can we find $M$?

II Minkowski Problem:

Given a strictly positive real function $f$ defined on $S^2$, find a strictly convex compact surface $M \subset \mathbb{R}^3$ s.t. $K(M)$ at the point $x$ equals $f(n(x))$ where $n(x)$ is the normal to $M$ at $x$. Then the PDE is

$$\frac{u_{xx}u_{yy} - u_{xy}^2}{(1+|\nabla u|^2)} = f(n(x)) = g(\nabla u)$$

These PDEs are of the Monge-Ampère type which is

$$L(u) = A(u_{xx}u_{yy} - u_{xy}^2) + Bu_{xx} + Cu_{xy} + Du_{yy} + E = 0$$

where $A$ $B$ $C$ $D$ and $E$ are functions depending on $x, y, u, u_x$ and $u_y$ only

Ellipticity result:

if $\bar{x}$ is a variable with values in a domain $\Omega \subset \mathbb{R}^n$ and $f(\bar{x}, u, D^2u)$ is a positive function

Then

$$L(u) = \det D^2u - f(\bar{x}, u, Du) = 0$$

is a nonlinear elliptic PDE if we restrict our solutions to be convex $\Rightarrow$ II is a nonlinear elliptic PDE.