VANDERBILT UNIVERSITY
MATH 234 — INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS
ASSIGNMENT 1.

Due date: Friday, Jan 18[1] .

It was mentioned in class that the features of a PDE change considerably depending on whether it is linear or non-linear. The goal of this assignment is to explore this difference.

The general form of a PDE for a function u of n variables, $u = u(x^1, x^2, \dots, x^n)$, is

$$F(x^1, x^2, \dots, x^n, u_{x^1}, u_{x^2}, \dots, u_{x^n}, u_{x^1x^1}, u_{x^2x^2}, \dots) = 0. \tag{1}$$

We saw in class that a PDE is called *linear* if the function F above is a linear[2] function of the unknown u and its derivatives; otherwise it is called *non-linear*.

Problem 1. For each one of the PDEs below, identify the function F such that the PDE can be written as in equation (1).

(a) heat equation in two spatial dimensions:

$$u_t - k(u_{xx} + u_{yy}) = 0.$$

(b) the one-dimensional wave equation:

$$u_{tt} - c^2 u_{xx} = 0.$$

(c) Laplace's equation in three-dimensions:

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

(d) Laplace's equation in n-dimensions:

$$\Delta u = 0.$$

(e) non-linear wave equation in one spatial dimension:

$$u_{tt} - c^2 u_{xx} = u^2.$$

Problem 2. Let u_1 and u_2 be two solutions of the heat equation in two spatial dimensions. Show that for any real numbers a and b, the function $v = au_1 + bu_2$ is also a solution. Formulate, and

[1] If you prefer, you can turn in this assignment in class on Thursday. If you hand it in on Friday, please bring it to my office. If you do not find me there, slide your assignment under the door.

[2] The concept of a linear function is covered in the courses which are pre-requisites for MATH 234. If you do not remember what a linear function is, review the material of such courses, or come to see me during office hours.

prove, a similar statement for the equations in (b), (c) and (d) of problem 1. Finally, show that in general a similar statement is not true for the non-linear wave equation in (e) of problem 1.

Problem 3. Determine whether the equations below are linear or non-linear.

(a) $u_{xx}^2 + u - x = 0$.

(b) $u_{tt} + u_{xx} + u_x + u_y + u = 0$.

(c) $u_{xxx} - u_{ttt} = x^2 \cos(x+t)$.

Problem 4. Consider now the two dimensional heat equation with a non-zero term on the right-hand side, i.e.,

$$u_t - k(u_{xx} + u_{yy}) = f,$$

where f is a function of x and y which is not identically zero. If u_1 and u_2 are two solutions of the above equation, is $v = au_1 + bu_2$ also a solution?

Problem 5. Given any PDE, let u_1 and u_2 be two solutions. From what you learned from the previous problems, what can you expect about the behavior of the function $v = au_1 + bu_2$, where a and b are arbitrary real numbers?