MATH 2300-04
TEST 2 – SOLUTIONS

VANDERBILT UNIVERSITY

Directions: Make sure you clearly indicate the pages where your solutions are written. Answers without justification will receive little or no credit. Write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

The notation and conventions in this exam are the same as used in class, unless stated otherwise.

If you need to use a theorem or formula that was stated in class, you do not need to prove it, unless a question explicitly says so. You do need, however, to state the theorems or formulas you invoke.

If you do not understand a question, or think that some problem is ambiguous, missing information, or incorrectly stated, write how you interpret the problem and solve it accordingly.

Question	Points
1 (20 pts)	
2 (20 pts)	
3 (20 pts)	
4 (20 pts)	
5 (20 pts)	
Extra Credit (5 pts)	
Cheat sheet (1 pts)	
TOTAL (100 pts)	

Question 1. (20 pts) Calculate the limit, or show that the limit does not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+2y^2}$.

(b) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

Solution 1. (a) We have

$$0 \le \left| \frac{5x^2y}{x^2+2y^2} \right| \le \frac{x^2}{x^2+2y^2} 5|y| \le 5|y| \to 0 \text{ as } y \to 0,$$

thus the limit is zero by the squeeze theorem.

(b) Along the y-axis, the limit is zero. Along the line $x = y$, the limit is $1/2$. Thus the limit does not exist.

Question 2. (20 pts) Let $f = f(x, y)$ be defined in the domain $D = \{(x, y) \in \mathbb{R}^2 \,|\, x \geq 0\}$. Let $u = u(r, s)$ and $v = v(r, s)$ be defined in $\mathbb{R}^2$. Suppose that f, u, v, and their partial derivatives up to order two are continuous. Suppose that

- For any $s \in \mathbb{R}$, $u(r, s) \geq 0$ if $r \geq 0$ and $u(r, s) < 0$ if $r < 0$.
- $v(r, s) \geq 0$ for any $r \in \mathbb{R}$ and any $s \in \mathbb{R}$.
- $u(1, 1) = 3$, $u(1, 2) = 1$, $v(1, 1) = 2$, $v(1, 2) = 1$.
- $f_x(3, 2) = -1$, $f_y(3, 2) = 1$, $f_x(1, 2) = 0$, $f_y(1, 2) = -2$.
- $u_r(1, 1) = 2$, $u_s(1, 1) = 1$, $u_r(1, 2) = 2$, $u_s(1, 2) = -1$
- $v_r(1, 1) = 1$, $v_s(1, 1) = 3$, $v_r(1, 2) = -1$, $v_s(1, 2) = -2$.

Let $g(r, s) = f(u(r, s), v(r, s))$.

(a) Determine the domain of g.

Hint: Draw a picture of $\mathbb{R}^2$ highlighting the domain of f. Then, draw another picture of $\mathbb{R}^2$ highlighting the region in the domain of u that is mapped into the domain of f. Do something similar for v. Do not overthink this problem. It should be a simple matter of understanding such pictures.

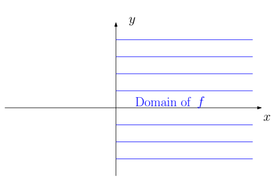
In (b) and (c) below, determine whether the above information is enough to find the given partial derivatives and, in case yes, find it.

(b) $g_s(1, 1)$.

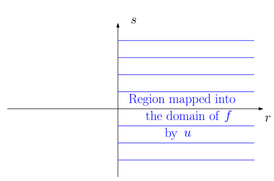
(c) $g_r(1, 2)$.

(d) Write an expression for g_{rs} in terms of partial derivatives of f, u, and v.

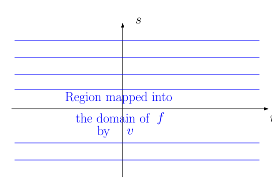
Solution 2. (a) Consider:



Domain of f



Region mapped into the domain of f by u



Region mapped into the domain of f by v

We see that the domain of g is $r \geq 0$.

(b) By the chain rule

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Since $g(r, s) = f(u(r, s), v(r, s))$, we can write this more explicitly as

$$\frac{\partial g(r,s)}{\partial s} = \left.\frac{\partial f(x,y)}{\partial x}\right|_{x=u(r,s),y=v(r,s)} \frac{\partial u(r,s)}{\partial s} + \left.\frac{\partial f(x,y)}{\partial y}\right|_{x=u(r,s),y=v(r,s)} \frac{\partial v(r,s)}{\partial s}$$

Plugging $(r,s)=(1,1)$ we find

$$\begin{aligned}\left.\frac{\partial g(r,s)}{\partial s}\right|_{(r,s)=(1,1)} &= \left.\frac{\partial f(x,y)}{\partial x}\right|_{x=u(1,1),y=v(1,1)} \left.\frac{\partial u(r,s)}{\partial s}\right|_{(r,s)=(1,1)} \\ &\quad + \left.\frac{\partial f(x,y)}{\partial y}\right|_{x=u(1,1),y=v(1,1)} \left.\frac{\partial v(r,s)}{\partial s}\right|_{(r,s)=(1,1)} \\ &= \left.\frac{\partial f(x,y)}{\partial x}\right|_{x=3,y=2} \left.\frac{\partial u(r,s)}{\partial s}\right|_{(r,s)=(1,1)} \\ &\quad + \left.\frac{\partial f(x,y)}{\partial y}\right|_{x=3,y=2} \left.\frac{\partial v(r,s)}{\partial s}\right|_{(r,s)=(1,1)} \\ &= f_x(3,2)u_s(1,1)+f_y(3,2)v_s(1,1) \\ &= -1\cdot 1+1\cdot 3=2.\end{aligned}$$

(c) By the chain rule

$$\frac{\partial g}{\partial r}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

Since $g(r,s)=f(u(r,s),v(r,s))$, we can write this more explicitly as

$$\frac{\partial g(r,s)}{\partial r} = \left.\frac{\partial f(x,y)}{\partial x}\right|_{x=u(r,s),y=v(r,s)} \frac{\partial u(r,s)}{\partial r} + \left.\frac{\partial f(x,y)}{\partial y}\right|_{x=u(r,s),y=v(r,s)} \frac{\partial v(r,s)}{\partial r}$$

Plugging $(r,s)=(1,2)$ we find

$$\begin{aligned}\left.\frac{\partial g(r,s)}{\partial r}\right|_{(r,s)=(1,2)} &= \left.\frac{\partial f(x,y)}{\partial x}\right|_{x=u(1,2),y=v(1,2)} \left.\frac{\partial u(r,s)}{\partial r}\right|_{(r,s)=(1,2)} \\ &\quad + \left.\frac{\partial f(x,y)}{\partial y}\right|_{x=u(1,2),y=v(1,2)} \left.\frac{\partial v(r,s)}{\partial r}\right|_{(r,s)=(1,2)} \\ &= \left.\frac{\partial f(x,y)}{\partial x}\right|_{x=1,y=1} \left.\frac{\partial u(r,s)}{\partial r}\right|_{(r,s)=(1,2)} \\ &\quad + \left.\frac{\partial f(x,y)}{\partial y}\right|_{x=1,y=1} \left.\frac{\partial v(r,s)}{\partial r}\right|_{(r,s)=(1,2)} \\ &= f_x(1,1)u_r(1,2)+f_y(1,1)v_r(1,2) \\ &= f_x(1,1)\cdot 2+f_y(1,1)\cdot(-1),\end{aligned}$$

which cannot be computed since the derivatives of f at $(1,1)$ are not given.

(d) By the assumptions, $g_{rs}=g_{sr}$ and $f_{xy}=f_{yx}$. By successive applications of the chain rule,

$$\begin{aligned} g_r &= f_x u_r + f_y v_r, \\ g_{rs} &= (f_x)_s u_r + f_x u_{rs} + (f_y)_s v_r + f_y v_{rs} \\ &= (f_{xx}u_s + f_{xy}v_s)u_r + f_x u_{rs} + (f_{xy}u_s + f_{yy}v_s)v_r + f_y v_{rs}.\end{aligned}$$

Question 3. (20 pts) Consider the function

$$f(x,y) = 4xy^2 - x^2y^2 - xy^3.$$

Let D be the closed triangular region in the xy-plane with vertices $(0,0)$, $(0,6)$, and $(6,0)$.

(a) Explain why f has an absolute maximum value and an absolute minimum value in D.

(b) Find the absolute maximum value and the absolute minimum value of f in D.

Solution 3. (a) f is a continuous function and D is a closed bounded set. Hence f attains an absolute maximum and absolute minimum in D by the extreme value theorem.

(b) Compute $f_x(x,y) = 4y^2 - 2xy^2 - y^3 = 0$. This gives $y = 0$ or $y = 4 - 2x$. Since $y = 0$ belong to the boundary of D, which will be analyzed separately below, we can ignore it for now. Next, find $f_y(x,y) = 8xy - 2x^2y - 3xy^2$. Setting this expression equal to zero and plugging in $y = 4 - 2x$ we find

$$8x(4-2x) - 2x^2(4-2x) - 3x(4-2x)^2 = x(4-2x)(4x-4) = 0,$$

which gives $x = 0$, $x = 1$, or $x = 2$. As before, $x = 0$ is on the boundary of D so we can ignore it for now. When $x = 2$, we have $y = 4 - 2x = 0$ which can also be ignored at this point since y then belongs to the boundary of D. Hence, we are left with $x = 1$, so $y = 2$, and this gives $f(1,2) = 4$.

Now we analyze the behavior of f on the boundary of D. The boundary is given by three lines:

$$L_1 : \{x = 0,\ 0 \le x \le 6\},\ L_2 : \{(x,y) \,|\, y = -x + 6,\ 0 \le x \le 6\},\ L_3 : \{y = 0,\ 0 \le y \le 6\}.$$

Along L_1 and L_3 we have $f(x,0) = 0$ and $f(0,y) = 0$, respectively. Along L_2,

$$g(x) = f(x, 6-x) = -2(x^3 - 12x^2 + 36x).$$

We seek that absolute maximum and minimum of g on $[0,6]$. Computing $g'(x) = -6(x-2)(x-6) = 0$ we find $x = 2$ and $x = 6$. For $x = 2$ we have $y = 4$ and $f(2,4) = -64$. $x = 6$ is an endpoint, and the endpoints $x = 0$ and $x = 6$ have already been tested in L_3 and L_1, respectively.

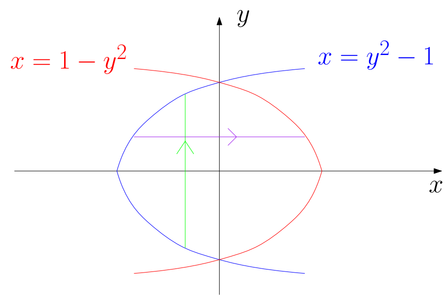
We conclude that the absolute maximum is 4, occurring at $(1,2)$, and the absolute minimum is -64, occurring at $(2,4)$.

Question 4. (20 pts) Let D be the region in the xy-plane bounded by the curves $x = y^2 - 1$ and $x = 1 - y^2$. Let f be a continuous function defined in D.

(a) Express $\iint\limits_D f(x, y)\, dA$ as iterated integrals, integrating first in x and then in y.

(b) Express $\iint\limits_D f(x, y)\, dA$ as iterated integrals, integrating first in y and then in x.

Solution 4. The region D is illustrated in the picture:



Purple line: integrating in x first; green line: integrating in y first.

The curves intersect at $(0, -1)$ and $(0, 1)$. The blue curve intersects the x-axis at $(-1, 0)$ and the red curve at $(1, 0)$.

(a) We have

$$\iint\limits_D f(x, y)\, dA = \int_{-1}^{1} \int_{y^2-1}^{1-y^2} f(x, y)\, dxdy.$$

(a) We have

$$\iint\limits_D f(x, y)\, dA = \int_{-1}^{0} \int_{-\sqrt{x+1}}^{\sqrt{x+1}} f(x, y)\, dydx + \int_{0}^{1} \int_{-\sqrt{1-x}}^{\sqrt{1-x}} f(x, y)\, dydx.$$

Question 5. (20 pts) Let $f(x, y, z) = xy + xz + yz$.

(a) Find ∇f.

(b) Suppose that $D_{\mathbf{u}} f(a, b, c)$ maximizes the rate of change of f at (a, b, c). Determine the direction of $\mathbf{u}$.

Solution 5. (a) We have

$$\nabla f(x, y, z) = (f_x, f_y, f_z) = (y + z, x + z, x + y).$$

(b) Since the directional derivative is maximized in the direction of the gradient, we have

$$\mathbf{u} = \frac{\nabla f(a, b, c)}{|\nabla f(a, b, c)|} = \frac{(b + c, a + c, a + b)}{\sqrt{(b + c)^2 + (a + c)^2 + (a + b)^2}}.$$

Extra credit. (5 pts) Use $\varepsilon - \delta$ arguments to show that the function $f(x, y, z) = x^2 + y^2 + z^2$ is continuous at $(0, 0, 0)$.

Solution to the extra credit. Let $\varepsilon > 0$ be given. Then

$$|f(x, y, z) - f(0, 0, 0)| = x^2 + y^2 + z^2 = \left(\sqrt{x^2 + y^2 + z^2}\right)^2 < \varepsilon$$

whenever

$$|(x, y, z) - (0, 0, 0)| = \sqrt{x^2 + y^2 + z^2} < \delta = \sqrt{\varepsilon}.$$

Cheat sheet. (1 pts) Do not forget that you must upload your cheat sheet.