MATH 2300-04
PRACTICE TEST 2 – SOLUTIONS

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Directions: This practice test should be used as a study guide, illustrating the concepts that will be emphasized in the first test. This does not mean that the actual test will be restricted to the content of the practice. Try to identify, from the questions below, the concepts and methods that you should master for the test. For each question in the practice test, study the ideas and techniques connected to the problem, even if they are not directly used in your solution.

Take this also as an opportunity to practice how you will write your solutions in the test. For this, write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

The second test will cover the following sections of the textbook: 14.1, 14.2, 14.3, 14.4, 14.5, 14.6, 14.7, 15.1, 15.2 (14.8 will not be in the test).

Question 1. Calculate the limit of the given functions, or show that the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$.

(b) $\displaystyle\lim_{(x,y)\to(1,2)} \frac{xy-2x-y+2}{\sqrt{x^2-2x+1+3(y-2)^4}}$

Solution 1. (a) Along the x-axis, where $y=0$ we have that the limit is zero. Along $y=x^2$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0}\frac{x^4}{x^4+x^4} = \frac{1}{2},$$

so the limit does not exist.

(b) Write

$$0 \leq \left|\frac{xy-2x-y+2}{\sqrt{x^2-2x+1+3(y-2)^4}}\right| = \left|\frac{(x-1)(y-2)}{\sqrt{(x-1)^2+3(y-2)^4}}\right| \leq |y-2| \to 0$$

as $y\to 2$, so the limit is zero by the squeeze theorem.

Question 2. Let $f=f(x,y,z)$ and $u=u(r,s)$, $v=v(r,s)$, $w=w(r,s)$ be differentiable functions defined in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. Let $g(r,s)=f(u(r,s),v(r,s),w(r,s))$.

(a) What is the domain of g?

(b) Explain why g is differentiable on its domain.

(c) Assume that $u(0,0) = v(0,0) = w(0,0) = 1$, $u(1,1) = v(1,1) = w(1,1) = 0$, $f_x(1,1,1) = f_y(1,1,1) = 2$, $f_z(1,1,1) = 3$, $f_x(0,0,0) = f_y(0,0,0) = f_z(0,0,0) = 2$, $u_r(0,0) = v_r(0,0) = w_r(0,0) = -1$, and $u_s(1,1) = v_s(1,1) = w_s(1,1) = 1$. Determine whether this information suffices to compute $g_r(0,0)$. In case yes, find its value.

Solution 2. (a) Since both functions are defined for every value of their arguments, the domain of g is $\mathbb{R}^2$.

(b) g is the composition of two differentiable functions, hence differentiable.

(c) By the chain rule

$$\frac{\partial g}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}.$$

Since $g(r,s) = f(u(r,s), v(r,s), w(r,s))$, we can write this more explicitly as

$$\begin{aligned}\frac{\partial g(r,s)}{\partial r} = &\left.\frac{\partial f(x,y,z)}{\partial x}\right|_{x=u(r,s),y=v(r,s),z=w(r,s)} \frac{\partial u(r,s)}{\partial r}\\ &+ \left.\frac{\partial f(x,y,z)}{\partial y}\right|_{x=u(r,s),y=v(r,s),z=w(r,s)} \frac{\partial v(r,s)}{\partial r}\\ &+ \left.\frac{\partial f(x,y,z)}{\partial z}\right|_{x=u(r,s),y=v(r,s),z=w(r,s)} \frac{\partial w(r,s)}{\partial r}.\end{aligned}$$

Plugging $(r,s) = (0,0)$ we find

$$\begin{aligned}\left.\frac{\partial g(r,s)}{\partial r}\right|_{(r,s)=(0,0)} = &\left.\frac{\partial f(x,y,z)}{\partial x}\right|_{x=u(0,0),y=v(0,0),z=w(0,0)} \left.\frac{\partial u(r,s)}{\partial r}\right|_{(r,s)=(0,0)}\\ &+ \left.\frac{\partial f(x,y,z)}{\partial y}\right|_{x=u(0,0),y=v(0,0),z=w(0,0)} \left.\frac{\partial v(r,s)}{\partial r}\right|_{(r,s)=(0,0)}\\ &+ \left.\frac{\partial f(x,y,z)}{\partial z}\right|_{x=u(0,0),y=v(0,0),z=w(0,0)} \left.\frac{\partial w(r,s)}{\partial r}\right|_{(r,s)=(0,0)}\\ = &\left.\frac{\partial f(x,y,z)}{\partial x}\right|_{x=1,y=1,z=1} \left.\frac{\partial u(r,s)}{\partial r}\right|_{(r,s)=(0,0)}\\ &+ \left.\frac{\partial f(x,y,z)}{\partial y}\right|_{x=1,y=1,z=1} \left.\frac{\partial v(r,s)}{\partial r}\right|_{(r,s)=(0,0)}\\ &+ \left.\frac{\partial f(x,y,z)}{\partial z}\right|_{x=1,y=1,z=1} \left.\frac{\partial w(r,s)}{\partial r}\right|_{(r,s)=(0,0)}\\ = &f_x(1,1,1)u_r(0,0) + f_y(1,1,1)v_r(0,0) + f_z(1,1,1)w_r(0,0)\\ = &2\cdot(-1) + 2\cdot(-1) + 3\cdot(-1) = -7.\end{aligned}$$

Question 3. Let $f(x,y,z) = xe^{3yz}$, $\mathbf{u} = (2,-2,1)$.

(a) Find ∇f.

(b) Find the rate of change of f at the point $(3,0,1)$ in the direction of the vector $\frac{1}{3}\mathbf{u}$.

(c) In which direction is the rate of change of f the largest?

Solution 3. (a) $\nabla f(x,y,z) = (e^{3yz}, 3xze^{3yz}, 3xye^{3yz})$.

(b) $\nabla f(3,0,1) = (1,9,0)$. $D_{\frac{1}{3}\mathbf{u}} = \nabla f(3,0,1) \cdot \frac{1}{3}(2,-2,1) = \frac{1}{3}(1,9,0) \cdot (2,-2,1) = -\frac{16}{3}$.
(c) In the direction of $\frac{\nabla f}{|\nabla f|}$.

Question 4. Let D be the disk $D = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 4\}$. Explain why the function

$$f(x,y) = e^{-x^2-y^2}(x^2 + 2y^2)$$

has an absolute maximum and an absolute minimum value on D, and then find these values.

Solution 4. The function f is continuous and the region D is closed and bounded. Therefore, the extreme value theorem applies and f attains an absolute maximum and an absolute minimum in D.

Compute

$$f_x(x,y) = -2xe^{-x^2-y^2}(x^2 + 2y^2 - 1)$$

and

$$f_y(x,y) = -2ye^{-x^2-y^2}(x^2 + 2y^2 - 2).$$

Set $f_x = 0 = f_y$. From (4), we have $x = 0$ or $x^2 + 2y^2 - 1 = 0$. With $x = 0$, (4) gives $y = 0$ or $x^2 + 2y^2 - 2 = 2y^2 - 2 = 0$, thus $y = \pm 1$. Hence we obtain that $(0,0)$, $(0,1)$, and $(0,-1)$ are possible solutions. Next, with $x^2 + 2y^2 - 1 = 0$, we have $x^2 + 2y^2 - 2 \neq 0$, thus (4) gives $y = 0$. Then $x^2 + 2y^2 - 1 = x^2 - 1 = 0$ and thus $x = \pm 1$, and $(1,0)$ and $(-1,0)$ are possible solutions.

Compute

$$f(0,0) = 0,\ f(0,1) = 2e^{-1} = f(0,-1),\ f(1,0) = e^{-1} = f(-1,0).$$

We now consider the values of f on the boundary of the domain, i.e., $x^2 + y^2 = 4$, so $y^2 = 4 - x^2$, $-2 \leq x \leq 2$. Using $y^2 = 4 - x^2$ to eliminate y from $f(x,y)$, we find that on the boundary of D, f reduces to g, where

$$g(x) = e^{-4}(-x^2 + 8).$$

The function g is a parabola and we immediately see that it has a maximum value when $x = 0$ and a minimum value when $x = \pm 2$, giving, respectively, $y = \pm 2$ and $y = 0$. Compute

$$f(0,2) = 8e^{-4} = f(0,-2),\ f(2,0) = 4e^{-4} = f(-2,0).$$

Since $8e^{-4} < 2e^{-1}$, we conclude that that the maximum value is $2e^{-1}$. The minimum value is 0.

Question 5. Let D be the region in $\mathbb{R}^2$ bounded by the curves $x - y - 42 = 0$ and $x - y^2 = 0$. Let $f(x,y) = e^{xy}$.

(a) Express $\iint_D f(x,y)\,dA$ as iterated integrals, integrating first in x and then in y.

(b) Express $\iint_D f(x,y)\,dA$ as iterated integrals, integrating first in y and then in x.

Solution 5. The intersection of the curves happens at $(36,-6)$ and $(49,7)$. Thus

(a) $\displaystyle\iint_D f(x,y)\,dA = \int_{-6}^{7}\int_{y^2}^{y+42} e^{xy}\,dxdy.$

(b) $\displaystyle\iint\limits_D f(x,y)\,dA = \int_0^{36}\int_{-\sqrt{x}}^{\sqrt{x}} e^{xy}\,dydx + \int_{36}^{49}\int_{x-42}^{\sqrt{x}} e^{xy}\,dydx.$

Question 6. For each question above, understand which concept or theorem is needed to justify your answers and the steps you used to solve the problems. For example, suppose that you have to go over every step in your solutions and provide a justification of why it is true.