MATH 2300-04
EXTRA CREDIT ASSIGNMENT

VANDERBILT UNIVERSITY

Question 1. Show that the normal vector $\mathbf{N}$ to a curve points toward the concave side of the curve.

Solution 1. Let $\mathbf{r}$ be a smooth curve defined on an interval (a, b) and let $t_0 \in (a, b)$. Let P be the plane determined by $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$. To show that $\mathbf{N}(t_0)$ points to the concave side of the curve it suffices to show that the projection of $\mathbf{N}(t_0)$ onto P points to the concave side of the curve obtained by projecting $\mathbf{r}$ onto P. Hence, we can assume that $\mathbf{r}$ is a planar curve in the neighborhood of $\mathbf{r}(t_0)$. Without loss of generality we can assume that $\mathbf{T}(t_0)$ points in the same direction and is parallel to the x-axis, and that $\mathbf{r}(t_0) = (0, 0)$. Since the curve is assumed to have a well-defined concavity near $\mathbf{r}(t_0)$, we can also assume that the curve lies below the y-axis in the neighborhood of the origin (see figure 1).

Since

$$\mathbf{N}(t_0) = \lim_{h \to 0} \frac{\mathbf{T}(t_0 + h) - \mathbf{T}(t_0)}{h},$$

it suffices to show that the vector $(\mathbf{T}(t_0 + h) - \mathbf{T}(t_0))/h$ points to the concave side of the curve for every h sufficiently small. Then, since the curve lies below the y-axis and $\mathbf{T}(t_0)$ has zero y-coordinate, we have that $\mathbf{T}(t_0 + h) - \mathbf{T}(t_0)$ has negative y-coordinate for $h > 0$ and positive y-coordinate for $h < 0$. It follows that $(\mathbf{T}(t_0 + h) - \mathbf{T}(t_0))/h$ has negative y-coordinate in both cases. Thus, $(\mathbf{T}(t_0 + h) - \mathbf{T}(t_0))/h$ points to the concave side of the curve (if h is sufficiently small), and so does $\mathbf{N}(t_0)$.

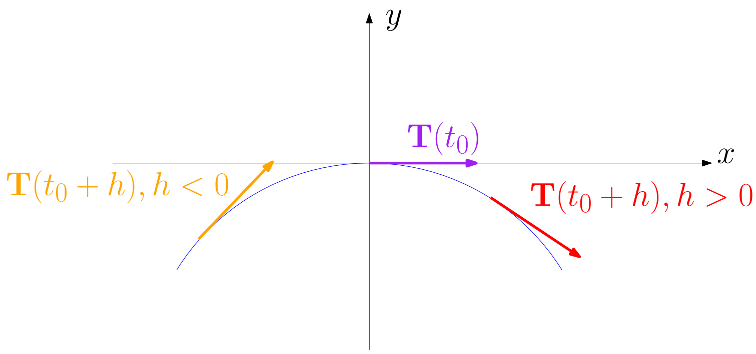

FIGURE 1. Problem 1.

Question 2. Show that Kepler's laws of planetary motion are a consequence of Newton's laws of motion and gravitation (see section 13.4 of the textbook).

Solution 2. This is done in section 13.4 of the textbook.