VANDERBILT UNIVERSITY, MATH 2300-04, F 20
EXAMPLES OF SECTION 16.5

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Question 1. Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field in $\mathbb{R}^2$. Assume that P and Q have continuous partial derivatives. Considering $\mathbf{F}$ as a vector field in $\mathbb{R}^3$, show that

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint\limits_{D} \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA, \tag{1}$$

where D is a region in the xy-plane.

Solution 1. Write $\mathbf{F}$ as $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + 0\,\mathbf{k}$, so that it is defined in $\mathbb{R}^3$. Notice that P and Q do not depend on the z-variable since they were originally defined in $\mathbb{R}^2$. Compute

$$\operatorname{curl} \mathbf{F} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{bmatrix} = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,\mathbf{k},$$

where we used that $\frac{\partial Q}{\partial z} = \frac{\partial P}{\partial z} = 0$. Then

$$\operatorname{curl} \mathbf{F} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

On the other hand, Green's theorem gives

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint\limits_{D} \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dA,$$

so that

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint\limits_{D} \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA,$$

as desired.

Remark. Formula (1) is known as the vector form of Green's theorem.