VANDERBILT UNIVERSITY, MATH 2300-04, F 20
EXAMPLES OF SECTION 16.4

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Question 1. Use Green's theorem to show that the area of a planar region D is given by

$$A = \int_{\partial D} x\,dy = -\int_{\partial D} y\,dx = \frac{1}{2}\int_{\partial D}(x\,dy - y\,dx).$$

Question 2. Suppose that the vertices of a polygon, in counterclockwise order, are (x_1, y_1), (x_2, y_2), ..., (x_n, y_n). Show that the area of the polygon is

$$A = \frac{1}{2}\Big((x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)\Big).$$

Solution 1. We know that

$$A = \iint\limits_D dA.$$

To apply Green's theorem, we seek functions P and Q such that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

We can choose $P(x, y) = 0$ and $Q(x, y) = x$, in which case

$$A = \iint\limits_D dA = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_{\partial D}(P\,dx + Q\,dy) = \int_{\partial D} x\,dy.$$

Alternatively, we can also choose $P(x, y) = -y$ and $Q(x, y) = 0$, in which case

$$A = \iint\limits_D dA = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_{\partial D}(P\,dx + Q\,dy) = -\int_{\partial D} y\,dx.$$

Yet another possibility is $P(x, y) = -\frac{1}{2}y$ and $Q(x, y) = \frac{1}{2}x$, so that

$$A = \iint\limits_D dA = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_{\partial D}(P\,dx + Q\,dy) = \frac{1}{2}\int_{\partial D}(x\,dy - y\,dx).$$

Solution 2. Consider two consecutive vertices (x_i, y_i) and (x_{i+1}, y_{i+1}). Let C_i be the line segment joining (x_i, y_i) to (x_{i+1}, y_{i+1}), $i = 1, \ldots, n$, with the convention that $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. Let us compute

$$\int_{C_i}(x\,dy - y\,dx).$$

To do so, we can parametrize C_i as

$$\mathbf{r}_i = \big((1-t)x_i + tx_{i+1}\big)\mathbf{i} + \big((1-t)y_i + ty_{i+1}\big)\mathbf{j},\ 0 \le t \le 1.$$

Then, in parametric form,

$$x = (1-t)x_i + tx_{i+1},$$

so that

$$dx = (x_{i+1} - x_i)\,dt,$$

and

$$y = (1-t)y_i + ty_{i+1},$$

so that

$$dy = (y_{i+1} - y_i)\,dt.$$

Therefore,

$$\begin{aligned}\int_{C_i} (x\,dy - y\,dx) &= \int_0^1 \Big(((1-t)x_i + tx_{i+1})(y_{i+1} - y_i) + ((1-t)y_i + ty_{i+1})(x_{i+1} - x_i)\Big)\,dt \\ &= \int_0^1 (x_i y_{i+1} - x_{i+2} y_i)\,dt \\ &= x_i y_{i+1} - x_{i+1} y_i.\end{aligned} \tag{1}$$

Denoting by D the region enclosed by the polygon, from question 1 we have

$$A = \frac{1}{2}\int_{\partial D} (x\,dy - y\,dx) = \frac{1}{2}\sum_{i=1}^{n} \int_{C_i} (x\,dy - y\,dx).$$

Using (1) yields the desired result.