VANDERBILT UNIVERSITY, MATH 2300-04, F 20
EXAMPLES OF SECTION 15.7

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Question 1. Using cylindrical coordinates, find the mas of a ball B given by $x^2+y^2+z^2 \leq a^2$ if the density at any point is proportional to its distance from the z-axis.

Solution 1. The density is given by $\varrho(x, y, z) = K\sqrt{x^2 + y^2}$, where K is a constant. The mass is given by

$$m = \iiint_B \varrho(x, y, z) dV.$$

In cylindrical coordinates,

$$B = \{(r, \theta, z) \,|\, 0 \leq 0 \leq 2\pi, 0 \leq r \leq a, -\sqrt{a^2 - r^2} \leq z \leq \sqrt{a^2 - r^2}\},$$

so that

$$\begin{aligned}
m &= \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} \varrho(r\cos\theta, r\sin\theta, z) r \, dzdrd\theta \\
&= 2\int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2-r^2}} Kr^2 \, dzdrd\theta \\
&= 2\int_0^{2\pi} \int_0^a Kr^2\sqrt{a^2 - r^2} \, drd\theta \\
&= 2K \int_0^{2\pi} [\frac{1}{8}r(2r^2 - a^2)\sqrt{a^2 - r^2} + \frac{1}{8}a^4 \arcsin(\frac{r}{a})]_{r=0}^{r=a} \, d\theta \\
&= K\frac{\pi}{8}a^4 \int_0^{2\pi} d\theta \\
&= \frac{1}{4}a^4\pi^2 K.
\end{aligned}$$