VANDERBILT UNIVERSITY, MATH 2300-04, F 20
EXAMPLES OF SECTION 14.7

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Question 1. Find the absolute maximum and minimum values of $f(x,y) = 2x^3 + y^4$ on the region $D = \{(x,y) \,|\, x^2 + y^2 \leq 1\}$.

Solution 1. Notice that the assumptions of the extreme value theorem are satisfied, thus we know that f attains an absolute maximum and and absolute minimum in D.

We find $f_x(x,y) = 6x^2$ and $f_y(x,y) = 4y^3$, so $f_x(x,y) = 0 = f_y(x,y)$ gives $x = y = 0$, and $f(0,0) = 0$.

Next, we analyze the values of f on the boundary of D, which corresponds to $x^2 + y^2 = 1$, or

$$y^2 = 1 - x^2. \tag{1}$$

Thus, on the boundary,

$$f(x,y) = 2x^3 + (1-x^2)^2 = x^4 + 2x^3 - 2x^2 + 1,$$

where $-1 \leq x \leq 1$. Thus, we seek to find the maxima and minima of $g(x) = x^4 + 2x^3 - 2x^2 + 1$ on the interval $[-1, 1]$. Compute

$$g'(x) = 4x^3 + 6x^2 - 4x.$$

Setting $g'(x) = 0$ gives $x = 0$, $x = -2$, or $x = \frac{1}{2}$. From (1) we have the corresponding y values $y = \pm 1$, $y = -3$, and $y = \pm\frac{\sqrt{3}}{2}$, hence we need to check the values of f at $(0, \pm 1)$ and $(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$. Notice that $(-2,-3) \notin D$. The function g can have a maximum or minimum at the endpoints $x = \pm 1$, corresponding to $y = 0$, hence we also need to check f at $(1,0)$ and $(-1,0)$.

We find $f(0,\pm 1) = 1$, $f(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) = \frac{13}{16}$, $f(1,0) = 2$, and $f(-1,0) = -2$. Therefore, the absolute maximum is 2, occurring at $(1,0)$, and the absolute minimum is -2, occurring at $(-1,0)$.