VANDERBILT UNIVERSITY, MATH 2300-04, F 20
EXAMPLES OF SECTION 12.3

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Question 1. Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be non-zero vectors in $\mathbb{R}^3$. Define

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, \\
\mathbf{u}_2 &= \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1, \\
\mathbf{u}_3 &= \mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2.
\end{aligned}$$

Show that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ if $i \neq j$.

Solution 1. Let us compute,

$$\begin{aligned}
\mathbf{u}_1 \cdot \mathbf{u}_2 &= \mathbf{u}_1 \cdot (\mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1) \\
&= \mathbf{u}_1 \cdot \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 \cdot \mathbf{u}_1 \\
&= \mathbf{u}_1 \cdot \mathbf{v}_2 - \mathbf{u}_1 \cdot \mathbf{v}_2 \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{u}_1 \cdot \mathbf{u}_3 &= \mathbf{u}_1 \cdot (\mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2) \\
&= \mathbf{u}_1 \cdot \mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 \cdot \mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\underbrace{\mathbf{u}_1 \cdot \mathbf{u}_2}_{=0} \\
&= \mathbf{u}_1 \cdot \mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 \cdot \mathbf{u}_1 \\
&= \mathbf{u}_1 \cdot \mathbf{v}_3 - \mathbf{u}_1 \cdot \mathbf{v}_3 \\
&= 0.
\end{aligned}$$

Next,

$$\begin{aligned}
\mathbf{u}_3 \cdot \mathbf{u}_2 &= (\mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2) \cdot \mathbf{u}_2 \\
&= \mathbf{v}_3 \cdot \mathbf{u}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\underbrace{\mathbf{u}_1 \cdot \mathbf{u}_2}_{=0} - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 \cdot \mathbf{u}_2 \\
&= \mathbf{v}_3 \cdot \mathbf{u}_2 - \mathbf{u}_2 \cdot \mathbf{v}_3 \\
&= 0.
\end{aligned}$$