VANDERBILT UNIVERSITY, MATH 2300-04, F 20
EXAMPLES OF SECTION 12.1

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Question 1. Describe in words the regions of $\mathbb{R}^3$ determined by each one of the following equations, and then find the region defined by all the equations combined.

$$x = -y, \tag{1}$$
$$x^2 + y^2 = 25, \tag{2}$$
$$x^2 + y^2 + z^2 = 25. \tag{3}$$

Solution 1. Equation (1) is a plane orthogonal to the xy-plane and forming a 135° angle with the x-axis. Equation (2) is a cylinder of radius 5 orthogonal to the xy-plane and centered at the origin. Equation (3) is a sphere of radius 5 and centered at the origin.

For the second part of the question, we first find the region defined by equations (1) and (2), i.e., those points that satisfy simultaneously both equations. To do this, we use equation (2) into (1):

$$x^2 + y^2 = x^2 + (-x)^2 = 25 \Rightarrow x = \pm\sqrt{\frac{25}{2}}.$$

Using $x = \sqrt{\frac{25}{2}}$ into $y = -x$ we find $y = -\sqrt{\frac{25}{2}}$. Using $x = -\sqrt{\frac{25}{2}}$ into $y = -x$ we find $y = \sqrt{\frac{25}{2}}$. Therefore, equations (1) and (2) together give two solutions: $x = \sqrt{\frac{25}{2}}$ and $y = -\sqrt{\frac{25}{2}}$, or $x = -\sqrt{\frac{25}{2}}$ and $y = \sqrt{\frac{25}{2}}$. Each one of these solutions determines a *line* in $\mathbb{R}^3$. For, regions in $\mathbb{R}^3$ are specified by three coordinates (x, y, z); the solution $x = \sqrt{\frac{25}{2}}$ and $y = -\sqrt{\frac{25}{2}}$ has z unspecified, so it corresponds to all points in $\mathbb{R}^3$ of the form $(\sqrt{\frac{25}{2}}, -\sqrt{\frac{25}{2}}, z)$, where $z \in \mathbb{R}$. This is a line orthogonal to the xy-plane and passing through $x = \sqrt{\frac{25}{2}}$ and $y = -\sqrt{\frac{25}{2}}$. Similarly for the solution $x = -\sqrt{\frac{25}{2}}$ and $y = \sqrt{\frac{25}{2}}$.

Next, we consider the points $(x, y, z) \in \mathbb{R}^3$ that satisfy all the equations (1), (2), and (3). We already know that one solution to equations (1) and (2) is $x = \sqrt{\frac{25}{2}}$ and $y = -\sqrt{\frac{25}{2}}$. Using these values into (3) gives

$$\frac{25}{2} + \frac{25}{2} + z^2 = 25 \Rightarrow z = 0,$$

so we get the point $(\sqrt{\frac{25}{2}}, -\sqrt{\frac{25}{2}}, 0)$. Similarly, using $x = -\sqrt{\frac{25}{2}}$ and $y = \sqrt{\frac{25}{2}}$ we find $(-\sqrt{\frac{25}{2}}, \sqrt{\frac{25}{2}}, 0)$. Hence, equations (1), (2), and (3) determine two points in $\mathbb{R}^3$, namely, $(\sqrt{\frac{25}{2}}, -\sqrt{\frac{25}{2}}, 0)$ and $(-\sqrt{\frac{25}{2}}, \sqrt{\frac{25}{2}}, 0)$.