VANDERBILT UNIVERSITY

MATH 2300 – MULTIVARIABLE CALCULUS

Test 2

NAME: Solutions.

Directions. This exam contains four questions. Make sure you clearly indicate the pages where your solutions are written. **Answers without justification will receive little or no credit.** Write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc).

Question	**Points**
1 (25 pts)	
2 (25 pts)	
3 (25 pts)	
4 (25 pts)	

Question 1. (25 pts). Let D be the region in $\mathbb{R}^3$ bounded by $x^2+y^2+z^2=1$, $x=0$, $z=0$, with $z \geq 0$ and $x \geq 0$.

(a) Express $\iiint\limits_D f(x,y,z)\,dV$ as an iterated integral.

(b) Using Cartesian, cylindrical, or spherical coordinates, evaluate the integral in (a) if $f(x,y,z) = \sqrt{x^2+y^2+z^2}$.

Solution 1. (a) The sphere $x^2+y^2+z^2=1$ and the planes $x=0$, $z=0$ are shown in figure 1. The region bounded by the surfaces with $x \geq 0$ and $z \geq 0$ is shown in figure 2.

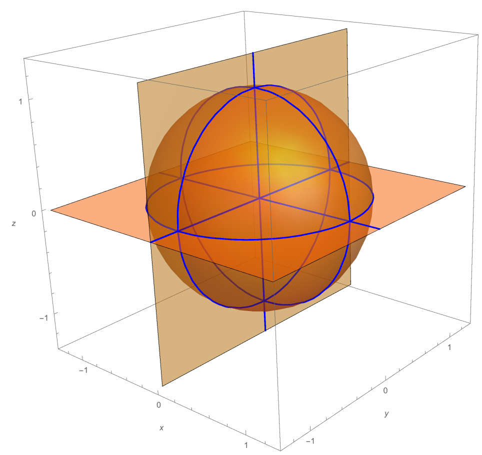

FIGURE 1. The sphere $x^2+y^2+z^2=1$ and the planes $x=0$, $z=0$.

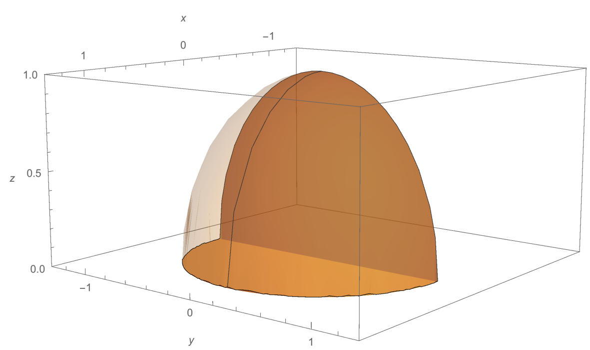

FIGURE 2. The region bounded by the sphere $x^2+y^2+z^2=1$ and the planes $x=0$, $z=0$ with $x \geq 0$ and $z \geq 0$.

Integrating in the order z, x, y we find

$$\iiint\limits_D f(x,y,z)\,dV = \int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dx\,dy.$$

Integrating in the order z, y, x we find

$$\iiint_D f(x,y,z)\,dV = \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx.$$

(b) In spherical coordinates $f(\rho,\phi,\theta) = \rho$. We find

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \sin\phi\,d\phi \int_0^1 \rho^3\,d\rho \\ &= \pi\,(-\cos\phi)\big|_0^{\frac{\pi}{2}} \left.\frac{\rho^4}{4}\right|_0^1 \\ &= \frac{\pi}{4}.\end{aligned}$$

Remark. Strictly speaking, we should have split the θ-integral from 0 to $\frac{\pi}{2}$ and from $\frac{3\pi}{2}$ to 2π, since the domain of θ is $[0, 2\pi]$. But it is easy to see that this will not change the answer.

Remark. This question is very similar to question 3 of the practice test. Students who understood that question should have encounter no difficulty here.

Question 2. (25 pts). Let R be the region in the first quadrant bounded by the curves $y = 3x$, $y = \frac{1}{x}$, $y = 2x$, and $y = \frac{3}{x}$.

(a) Find a change of variables $x = x(u, v)$, $y = y(u, v)$ that transforms the region R into a rectangle in the uv-plane.

(b) Using the transformation you found in part (a), evaluate the integral

$$\iint\limits_R \frac{y}{x}\, dA.$$

Solution 2. (a) The curves and the region R are shown in figure 3.

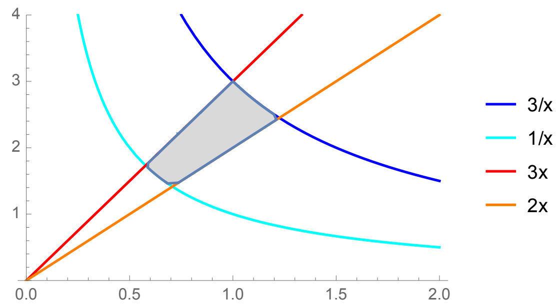

FIGURE 3. The region of question 2.

The curves can be written as $\frac{y}{x} = 3$, $xy = 1$, $\frac{y}{x} = 2$, and $xy = 3$. Set $u = xy$ and $v = \frac{y}{x}$. Then $uv = y^2$ and $\frac{u}{v} = x^2$. In these expressions we chose the positive square root because x and y are positive since they belong to the first quadrant. Thus we find

$$x = \sqrt{\frac{u}{v}} \text{ and } y = \sqrt{uv}.$$

The region in uv-coordinates is given by the rectangle $1 \le u \le 3$ and $2 \le v \le 3$.

(b) The Jacobian of the transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2v^{\frac{3}{2}}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{bmatrix} = \frac{1}{2v}.$$

The change of variable formula now gives

$$\iint\limits_R \frac{y}{x}\, dA = \iint\limits_S \frac{y(u, v)}{x(u, v)} \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA = \int_2^3 \int_1^3 v \frac{1}{2v}\, du\, dv = \frac{1}{2} \int_2^3 \int_1^3 du\, dv = 1.$$

Remark. Except for changing the numbers (say, $3x$ instead of $4x$), part (a) was the same question as problem 4(b) in the practice test. Part (b) was a straightforward application of the change of variables formula, which was also covered in question 5(a) of the practice. Students who understood those two problems should have encountered no difficulty here.

Question 3. (25 pts). Consider $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is a curve and $\mathbf{F}$ a vector field in $\mathbb{R}^3$.

(a) State the fundamental theorem of line integrals. Explain all your assumptions and your notation. Make a sketch that shows all quantities that you are using.

(b) Prove the fundamental theorem of line integrals. Explain how its hypotheses are used in the proof.

Solution 3. (a) Let C be a smooth curve given by the vector-valued function $\mathbf{r} : [a,b] \to \mathbb{R}^3$. Let f be a differentiable function of three variables whose gradient ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Figure 4 illustrate the curve C and the gradient of f along C.

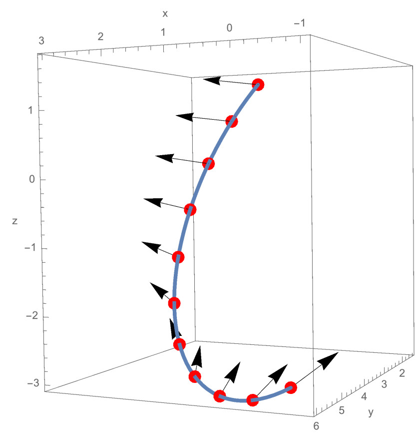

FIGURE 4. Curve and vector field in the fundamental theorem of line integrals.

(b) Writing $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, we have

$$\begin{aligned}\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\ &= \int_a^b \Big(\frac{\partial f(x(t),y(t),z(t))}{\partial x}x'(t) + \frac{\partial f(x(t),y(t),z(t))}{\partial y}y'(t) + \frac{\partial f(x(t),y(t),z(t))}{\partial z}z'(t)\Big)\,dt \\ &= \int_a^b \frac{d}{dt} f(x(t),y(t),z(t))\,dt \\ &= f(x(b),y(b),z(b)) - f(x(a),y(a),z(a)) = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).\end{aligned}$$

In the first equality we used the formula for the integral of vector fields along a smooth parametirzed curve. In the second equality we used the definition of the gradient and that $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$, along with the formula for the dot product. In the third equality we used the chain rule, which can be applied since f and $\mathbf{r}$ are differentiable by assumption. In the last equality we used the fundamental theorem of calculus, which can be invoked because the t-derivative of $f(x(t),y(t),z(t))$ is a continuous function in view of the assumptions on the gradient of f.

Remark. In question 10 of the practice test it had been indicated that a proof of one of the important theorems established in class was likely to be asked in the test.

Question 4. (25 pts). Let $\mathbf{F}$ be a vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$. Let S_2 be the square with vertices at $(-2,-2)$, $(-2,2)$, $(2,2)$, and $(2,-2)$, and S_3 be the square with vertices at $(-3,-3)$, $(-3,3)$, $(3,3)$, and $(3,-3)$. Let $C_2 = \partial S_2$ and $C_3 = \partial S_3$, both oriented counter-clockwise. Assume that:

- The functions P and Q have continuous partial derivatives in the region $x^2 + y^2 \geq 1$.
- $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ in the region outside S_2.
- $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$.

(a) Evaluate $\int_{C_3} \mathbf{F} \cdot d\mathbf{r}$.

(b) Is $\mathbf{F}$ a conservative vector field in the region $x^2 + y^2 \geq 1$?

(c) What can you say about the functions P and Q in the region $x^2 + y^2 \leq 1$?

Solution 4. The given curves are shown in figure 5.

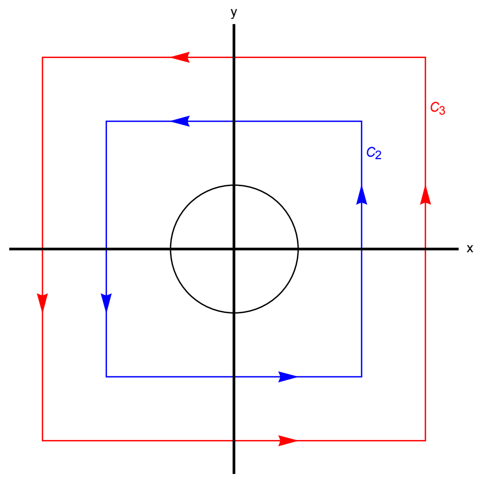

FIGURE 5. Curves C_2, C_3, and the circle of radius one.

(a) P and Q satisfy the assumption of Green's theorem in the region $x^2 + y^2 \geq 1$. Let D be the region between the curves C_2 and C_3. With the orientation convention of Green's theorem, D is on the left of the curve C_3 when it is traversed counter-clockwise, and on the left of $-C_2$, since C_2 has been initially oriented counter-clockwise (and in such case D is not on the left of C_2). Thus, $\partial D = (-C_2) \cup C_3$. Invoking Green's theorem and using the given information,

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0 = \int_{\partial D} (P\,dx + Q\,dy)$$

$$= \int_{-C_2} (P\,dx + Q\,dy) + \int_{C_3} (P\,dx + Q\,dy)$$

$$= -\int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}.$$

Thus,

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1.$$

(b) No, if it were then the integral on any closed curve in the region $x^2 + y^2 > 1$ would have to be zero, but $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$.

(c) We can make the following conditional statement. If P and Q satisfy the assumptions of Green's theorem in the region $x^2 + y^2 \leq 1$, then there must exist a point (x_0, y_0) inside S_2 such that $\frac{\partial Q(x_0,y_0)}{\partial x} \neq \frac{\partial P(x_0,y_0)}{\partial y}$. Otherwise we would be able to apply Green's theorem in the region inside S_2 to conclude that $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$, contradicting the given assumptions.

Remark. The trick to compute $\int_{C_3} \mathbf{F} \cdot d\mathbf{r}$ using $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ is the same employed in question 8(c) of the practice test (which in turn is the same as trick used in class to compute the integral of $-\frac{y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$ along an arbitrary curve enclosing the origin). Students who understood question 8(c) of the practice should have been able to easily solve (a). Part (b) relied on one of the several properties of conservative vector fields discussed in class, which students should have studied as indicated in question 10 of the practice test.