VANDERBILT UNIVERSITY

MATH 2300 – MULTIVARIABLE CALCULUS

Test 1

NAME: Solutions.

Directions. This exam contains six questions and an extra credit question. Make sure you clearly indicate the pages where your solutions are written. Answers without justification will receive little or no credit. Write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc). Notice that different questions may be worth different amounts of points.

Question	**Points**
1 (15 pts)	
2 (15 pts)	
3 (20 pts)	
4 (20 pts)	
5 (20 pts)	
6 (10 pts)	
Extra credit (10 pts)	
TOTAL	

Question 1. (15 pts). Sketch the given curves.

(a) $\mathbf{r}(t) = \langle (1-e^{-t})\cos t, (1-e^{-t})\sin t, e^{-t} \rangle$, $t \geq 0$.

(b) $\mathbf{r}(t) = \langle \cos(2t), \frac{\sin t}{2}, \cos t \rangle$.

Solution 1. (A description of the curves is given; drawing them by hand can then be easily done.)
(a) Compute $x^2 + y^2 = (1-e^{-t})^2$, so sections on the xy-plane correspond to circles that increase from radius zero to radius one as t increases. z changes from one to zero as t increases, so the curve is a spiral.
(b) Using $\cos(2t) = \cos^2 t - \sin^2 t$ we see that $x = z^2 - 4y^2$. Thus the curve lies on the saddle-like surface $x = z^2 - 4y^2$, whose sections $k = z^2 - 4y^2$ on the yz-plane are hyperbolas intersecting the y-axis for $k < 0$ and intersecting the z-axis for $k > 0$.

Question 2. (15 pts). Find an equation for the plane through the points $(1,1,1)$, $(0,0,2)$, and $(3,1,1)$.

Solution 2. Let $\mathbf{u} = \langle 1,1,1\rangle - \langle 0,0,2\rangle = \langle 1,1,-1,\rangle$ and $\mathbf{v} = \langle 3,1,1\rangle - \langle 0,0,2\rangle = \langle 3,1,-1,\rangle$. A normal to the plane is given by $\mathbf{u}\times\mathbf{v} = \langle 0,-2,-2\rangle$. Thus an equation for the plane is

$$(\langle x,y,z\rangle - \langle 0,0,2\rangle)\cdot\langle 0,-2,-2\rangle = 0.$$

Question 3. (20 pts). Find the limit or show that it does not exist:

$$\lim_{(x,y)\to(1,2)} \frac{xy-2x-y+2}{\sqrt{x^2-2x+1+3(y-2)^4}}.$$

Solution 3. We have

$$\frac{xy-2x-y+2}{\sqrt{x^2-2x+1+3(y-2)^4}} = \frac{(x-1)(y-2)}{\sqrt{(x-1)^2+3(y-2)^4}},$$

thus

$$\begin{aligned} 0 \le \left| \frac{xy-2x-y+2}{\sqrt{x^2-2x+1+3(y-2)^4}} \right| &= \left| \frac{(x-1)(y-2)}{\sqrt{(x-1)^2+3(y-2)^4}} \right| \\ &= \frac{|x-1||y-2|}{\sqrt{(x-1)^2+3(y-2)^4}} \\ &\le |y-2| \to 0 \;\text{ as }\; (x,y) \to (0,0). \end{aligned}$$

Therefore, by the squeeze theorem, the limit is zero.

Question 4. (20 pts). State the extreme value theorem. Discuss, in your own words, why it is true. Does the conclusion of the extreme value theorem hold if any of its hypotheses is not satisfied? Justify your answer.

Solution 4. This was done in great detail in class. Check you class notes from Sept 30 and Oct 3.

Question 5. (20 pts). Let D be the disk $D = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 4\}$. Explain why the function

$$f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$$

has an absolute maximum and an absolute minimum value on D, and then find these values.

Solution 5. The function f is continuous and the region D closed and bounded. Therefore, the extreme value theorem applies and f attains an absolute maximum and an absolute minimum in D.

Compute

$$f_x(x, y) = -2xe^{-x^2-y^2}(x^2 + 2y^2 - 1) \quad (1)$$

and

$$f_y(x, y) = -2ye^{-x^2-y^2}(x^2 + 2y^2 - 2). \quad (2)$$

Set $f_x = 0 = f_y$. From (1), we have $x = 0$ or $x^2 + 2y^2 - 1 = 0$. With $x = 0$, (2) gives $y = 0$ or $x^2 + 2y^2 - 2 = 2y^2 - 2 = 0$, thus $y = \pm 1$. Hence we obtain that $(0, 0)$, $(0, 1)$, and $(0, -1)$ are possible solutions. Next, with $x^2 + 2y^2 - 1 = 0$, we have $x^2 + 2y^2 - 2 \neq 0$, thus (2) gives $y = 0$. Then $x^2 + 2y^2 - 1 = x^2 - 1 = 0$ and thus $x = \pm 1$, and $(1, 0)$ and $(-1, 0)$ are possible solutions.

Compute

$$f(0, 0) = 0, \; f(0, 1) = 2e^{-1} = f(0, -1), \; f(1, 0) = e^{-1} = f(-1, 0).$$

We now consider the values of f on the boundary of the domain, i.e., $x^2 + y^2 = 4$, so $y^2 = 4 - x^2$, $-2 \leq x \leq 2$. Using $y^2 = 4 - x^2$ to eliminate y from $f(x, y)$, we find that on the boundary of D, f reduces to g, where

$$g(x) = e^{-4}(-x^2 + 8).$$

The function g is a parabola and we immediately see that it has a maximum value when $x = 0$ and a minimum value when $x = \pm 2$, giving, respectively, $y = \pm 2$ and $y = 0$. Compute

$$f(0, 2) = 8e^{-4} = f(0, -2), \; f(2, 0) = 4e^{-4} = f(-2, 0).$$

Since $8e^{-4} < 2e^{-1}$, we conclude that that the maximum value is $2e^{-1}$. The minimum value is 0.

Question 6. (10 pts). True or false? Justify your answer.

(a) If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, so is $\mathbf{u} \times \mathbf{v}$.

(b) If $f(x,y) \to L$ as $(x,y) \to (a,b)$ along every straight line through (a,b) then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

(c) If $\mathbf{r} : (a,b) \to \mathbb{R}^3$ is such that $|\mathbf{r}(t)| = 1$, for all $t \in (a,b)$, then $\mathbf{r}(t)$ lies on a circle of radius one.

Solution 6. (a) is false since $\mathbf{u} \times \mathbf{u} = \mathbf{0}$. (b) is false in light of the counter-example given in the the practice test. (c) is false; $|\mathbf{r}(t)| = 1$ only implies that the curve lies on a sphere of radius one.

Extra credit. (10 pts). Using $\varepsilon - \delta$, prove that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0,$$

where

$$f(x,y) = x^2 + 2y^2.$$

Solution 7. Let $\varepsilon > 0$ be given. Consider the inequality

$$|f(x,y) - 0| = x^2 + 2y^2 < \varepsilon. \tag{3}$$

Since

$$\sqrt{x^2 + 2y^2} < \sqrt{2x^2 + 2y^2} = \sqrt{2}\sqrt{x^2 + y^2} = \sqrt{2}|(x,y) - (0,0)|,$$

we see that if $|(x,y) - (0,0)| < \frac{\sqrt{\varepsilon}}{\sqrt{2}}$, then (3) holds. Thus we can choose $\delta = \frac{\sqrt{\varepsilon}}{\sqrt{2}}$.