**VANDERBILT UNIVERSITY**

MATH 2300 – MULTIVARIABLE CALCULUS

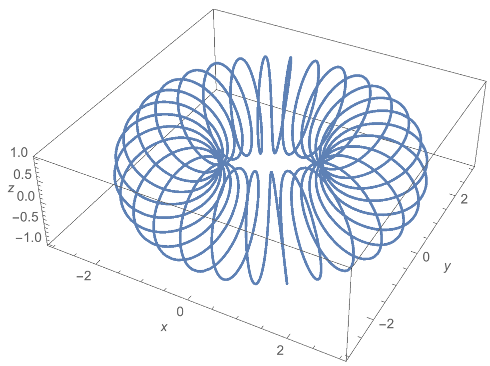
*Practice Test 1 — Solutions*

**Directions.** This practice test should be used as a study guide, illustrating the concepts that will be emphasized in the test. This does not mean that the actual test will be restricted to the content of the practice. Try to identify, from the questions below, the concepts and sections that you should master for the test. For each question in the practice test, study the ideas and techniques connected to the problem, even if they are not directly used in your solution.

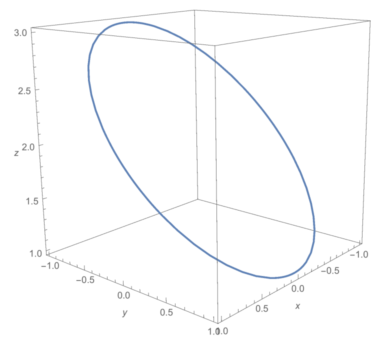
Take this also as an opportunity to practice how you will write your solutions in the test. For this, write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc).

**Question 1.** Match the curves below with the given graphs. Justify your answer.

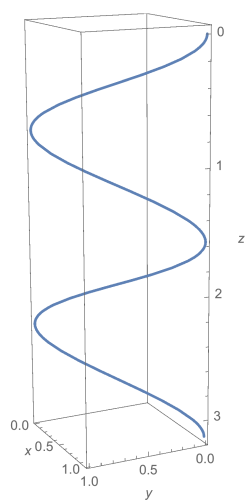
(a) $\mathbf{r}(t) = \langle (2+\sin(5t))\cos t, (2+\sin(5t))\sin t, \cos(5t)\rangle$
(b) $\mathbf{r}(t) = \langle \cos t, \sin t, 2-\sin t\rangle$
(c) $\mathbf{r}(t) = \langle \cos^2 t, \sin^2 t, \frac{1}{2}t\rangle$
(d) $\mathbf{r}(t) = \langle \cos t, \sin t, \cos(2t)\rangle$



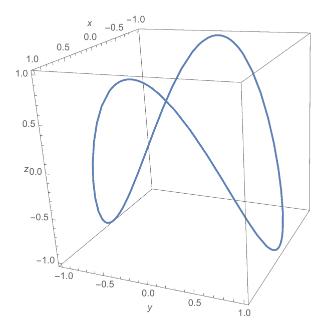

(I)




(II)




(III)




(IV)

**Solution 1.** For (a), we see that $-1 \le z \le 1$ and $-3 \le x, y \le 3$. Computing $x^2+y^2$, we find

$$x^2+y^2 = (2+\sin(5t))^2 \ge 1,$$

since $-1 \le \sin(5t) \le 1$. Thus the curve in (a) is such that is projection onto the $xy$ plane is always at a distant at least one from the origin, and we conclude that (a) corresponds to I.

For (b), $x^2+y^2=1$ and $z = 2-y$, so the curve lies at the intersection of the cylinder $x^2+y^2=1$ and and the plane $z = 2-y$. We conclude that (b) corresponds to II.

For (c), $x+y=1$ so the curve lies on the plane $y = 1-x$. We conclude that (c) corresponds to III.

For (d), $z = \cos(2t) = \cos^2 t - \sin^2 t = x^2 - y^2$, so the curve lies on the saddle-like surface $z = x^2 - y^2$. We conclude that (d) corresponds to IV.

**Question 2.** Where does the line through the points $(-3, 1, 0)$ and $(-1, 5, 6)$ intersect the plane $2x + y - z + 2 = 0$?

**Solution 2.** First we find the direction vector of the line: $\mathbf{v} = \langle -1, 5, 6\rangle - \langle -3, 1, 0\rangle = \langle 2, 4, 6\rangle$. Therefore the line through the given points can be written as

$$\mathbf{r}(t) = \langle -3, 1, 0\rangle + t\langle 2, 4, 6\rangle,$$

or yet $x(t) = -3 + 2t$, $y(t) = 1 + 4t$, $z(t) = 6t$. Plugging these equations into the equation of the plane,

$$2(-3 + 2t) + (1 + 4t) - 6t + 2 = 0,$$

gives $t = 3/2$, thus $(0, 7, 9)$.

**Question 3.** Calculate the limit of the given functions, or show that the limit does not exist.

(a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+2y^2}.$

(b) $\displaystyle \lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2}.$

**Solution 3.** For (a), notice that $x^2 \leq x^2+2y^2$ so that $0 \leq \frac{x^2}{x^2+2y^2} \leq 1$. Thus

$$0 \leq \left| \frac{5x^2y}{x^2+2y^2} \right| \leq 5|y| \to 0 \text{ as } y \to 0.$$

Hence, by the squeeze theorem

$$\lim_{(x,y)\to(0,0)} \left| \frac{5x^2y}{x^2+2y^2} \right| = 0,$$

and therefore

$$\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+2y^2} = 0.$$

For (b), write $f(x,y) = \frac{xy-y}{(x-1)^2+y^2}$. $f(x,0) = 0$ for $x \neq 1$, thus $f(x,0) \to 0$ as $x \to 1$. On the other hand, approaching $(1,0)$ along the line $y = x-1$, we find

$$f(x, x-1) = \frac{x(x-1)-(x-1)}{(x-1)^2+(x-1)^2} = \frac{1}{2},$$

and therefore $\frac{xy-y}{(x-1)^2+y^2} \to 1/2$ along the line $y = x-1$. We conclude that the limit does not exist.

**Question 4.**
(a) Let $F$ be a differentiable function of two variables $x$ and $y$, and suppose that $F(x, y) = 5$ defines $y$ implicitly as a differentiable function of $x$. Show that

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

(b) State the implicit function theorem. Explain, in words, why it is true. (*Hint:* use pictures to illustrate your argument.)

**Solution 4.** For (a), write $y = y(x)$ so that $F(x, y) = F(x, y(x))$ and apply the chain rule:

$$\frac{d}{dx}F(x, y(x)) = \frac{\partial F}{\partial x}(x, y(x))\frac{dx}{dx} + \frac{\partial F}{\partial y}(x, y(x))\frac{dy}{dx} = \frac{d}{dx}5 = 0,$$

which gives the result since $\frac{dx}{dx} = 1$.

(b) Implicit function theorem: If $F = F(x, y)$ is defined in a neighborhood of $(x_0, y_0)$, $F(x_0, y_0) = 0$, $F_x$ and $F_y$ exist and are continuous, and $F_y(x_0, y_0) \neq 0$, then $F(x, y) = 0$ defines $y$ as a function of $x$ in a neighborhood of $(x_0, y_0)$.

The basic intuition is the following. If $F_y(x_0, y_0) \neq 0$, then the tangent line to the curve $F(x, y) = 0$ at $(x_0, y_0)$ is not parallel to the $y$-axis. Since this tangent line approximates the curve near $(x_0, y_0)$, there does not exist a vertical line that intersects that curve $F(x, y) = 0$ twice in a small neighborhood of $(x_0, y_0)$. Hence, for each point $(x, y)$ on the curve (and near $(x_0, y_0)$) there corresponds a unique $y$, which means that $y$ can be viewed as a function of $x$ along $F(x, y) = 0$ and near $(x_0, y_0)$.

Similar statements hold for a function of three variables (see the textbook).

**Question 5.** Consider the function

$$f(x, y) = 4xy^2 - x^2y^2 - xy^3.$$

Let $D$ be the closed triangular region in the $xy$-plane with vertices $(0, 0)$, $(0, 6)$, and $(6, 0)$.

(a) Explain why $f$ has an absolute maximum value and an absolute minimum value in $D$.

(b) Find the absolute maximum value and the absolute minimum value of $f$ in $D$.

**Solution 5.** (a) $f$ is a continuous function and $D$ is a closed bounded set. Hence $f$ attains an absolute maximum and absolute minimum in $D$ by the extreme value theorem.

(b) Compute $f_x(x, y) = 4y^2 - 2xy^2 - y^3 = 0$. This gives $y = 0$ or $y = 4 - 2x$. Since $y = 0$ belong to the boundary of $D$, which will be analyzed separately below, we can ignore it for now. Next, find $f_y(x, y) = 8xy - 2x^2y - 3xy^2$. Setting this expression equal to zero and plugging in $y = 4 - 2x$ we find

$$8x(4 - 2x) - 2x^2(4 - 2x) - 3x(4 - 2x)^2 = x(4 - 2x)(4x - 4) = 0,$$

which gives $x = 0$, $x = 1$, or $x = 2$. As before, $x = 0$ is on the boundary of $D$ so we can ignore it for now. When $x = 2$, we have $y = 4 - 2x = 0$ which can also be ignored at this point since $y$ then belongs to the boundary of $D$. Hence, we are left with $x = 1$, so $y = 2$, and this gives $f(1, 2) = 4$.

Now we analyze the behavior of $f$ on the boundary of $D$. The boundary is given by three lines:

$$L_1 : \{x = 0,\ 0 \leq x \leq 6\},\ L_2 : \{(x, y) \,|\, y = -x + 6,\ 0 \leq x \leq 6\},\ L_3 : \{y = 0,\ 0 \leq y \leq 6\}.$$

Along $L_1$ and $L_3$ we have $f(x, 0) = 0$ and $f(0, y) = 0$, respectively. Along $L_2$,

$$g(x) = f(x, 6 - x) = -2(x^3 - 12x^2 + 36x).$$

We seek that absolute maximum and minimum of $g$ on $[0, 6]$. Computing $g'(x) = -6(x-2)(x-6) = 0$ we find $x = 2$ and $x = 6$. For $x = 2$ we have $y = 4$ and $f(2, 4) = -64$. $x = 6$ is an endpoint, and the endpoints $x = 0$ and $x = 6$ have already been tested in $L_3$ and $L_1$, respectively.

We conclude that the absolute maximum is 4, occurring at $(1, 2)$, and the absolute minimum is $-64$, occurring at $(2, 4)$.

**Question 6.** True or false? Justify your answer.

(a) If $f(x,y) \to L$ as $(x,y) \to (a,b)$ along every straight line through $(a,b)$ then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

(b) For any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, $(\mathbf{u}+\mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$.

(c) If $\mathbf{r} : (a,b) \to \mathbb{R}^3$ is differentiable, and $|\mathbf{r}(t)| = 3$ for all $t \in (a,b)$, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all $t \in (a,b)$.

**Solution 6.** (a) False. The function $f(x,y) = \frac{xy^2}{x^2+y^4}$ has limit 1/2 when $(x,y) \to (0,0)$ along the curve $x = y^2$, but the limit along any straight line through the origin is zero.

(b) True; compute both sides and compare.

(c) True: $\mathbf{r}(t) \cdot \mathbf{r}(t) = 9$, so differentiating gives $2\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$.