

VANDERBILT UNIVERSITY

MATH 2300 – MULTIVARIABLE CALCULUS

Examples of section 16.5

Question 1. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field in \mathbb{R}^2 . Assume that P and Q have continuous partial derivatives. Considering \mathbf{F} as a vector field in \mathbb{R}^3 , show that

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k} \, dA, \quad (1)$$

where D is a region in the xy -plane.

Solution 1. Write \mathbf{F} as $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + 0\mathbf{k}$, so that it is defined in \mathbb{R}^3 . Notice that P and Q do not depend on the z -variable since they were originally defined in \mathbb{R}^2 . Compute

$$\operatorname{curl} \mathbf{F} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{bmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k},$$

where we used that $\frac{\partial Q}{\partial z} = \frac{\partial P}{\partial z} = 0$. Then

$$\operatorname{curl} \mathbf{F} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

On the other hand, Green's theorem gives

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

so that

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k} \, dA,$$

as desired.

Remark. Formula (1) is known as the vector form of Green's theorem.