VANDERBILT UNIVERSITY

MATH 2300 – MULTIVARIABLE CALCULUS

Examples of section 12.4

Question 1. Show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ and that its direction is given by the right-hand rule. Assume that the vectors are non-zero and non-parallel.

Solution 1. Compute

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} &= \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= (u_2v_3 - u_3v_2)u_1 + (u_3v_1 - u_1v_3)u_2 + (u_1v_2 - u_2v_1)u_3 \\ &= \cancel{u_2v_3u_1} - \cancel{u_3v_2u_1} + \cancel{u_3v_1u_2} - \cancel{u_1v_3u_2} + \cancel{u_1v_2u_3} - \cancel{u_2v_1u_3} \\ &= 0.\end{aligned}$$

Similarly one shows that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$.

For the second part, given $\mathbf{u}$ and $\mathbf{v}$, consider their initial points at the same point in space. Define a Cartesian coordinate system such that the xy-plane contains $\mathbf{u}$ and $\mathbf{v}$ and such that the x-axis points along $\mathbf{u}$. Then $\mathbf{u} = \langle u_1, 0, 0 \rangle$ and $\mathbf{v} = \langle v_1, v_2, 0 \rangle$, and $u_1 > 0$. Therefore

$$\mathbf{u} \times \mathbf{v} = \langle 0, 0, u_1v_2 \rangle.$$

From this we conclude that if $v_2 > 0$ then $\mathbf{u} \times \mathbf{v}$ points in the positive z-direction, and if $v_2 < 0$ then $\mathbf{u} \times \mathbf{v}$ points in the negative z-direction, agreeing with the right-hand rule.