VANDERBILT UNIVERSITY

MATH 2300 – MULTIVARIABLE CALCULUS

Examples of section 12.3

Question 1. If an object is moved from a point P to a point Q, the **displacement vector** is $\mathbf{D} = \overrightarrow{PQ}$. When the object is moved due to the action of a constant force $\mathbf{F}$, the **work** done to move the object is defined by $W = \mathbf{F} \cdot \mathbf{D}$, measure in J (Joules) in SI units.

Suppose that a wagon is pulled a distance $10m$ along a horizontal path by a force of $50N$ making a 60° angle with the horizontal. Calculate the work done in the process.

Solution 1. We compute

$$\begin{aligned} W &= \mathbf{F} \cdot \mathbf{D} \\ &= |\mathbf{F}|\,|\mathbf{D}| \cos\theta \\ &= 50N\,10m \cos 60^\circ \\ &= 250J. \end{aligned}$$

Question 2. Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be non-zero vectors in $\mathbb{R}^3$. Define

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{v}_1, \\ \mathbf{u}_2 &= \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1, \\ \mathbf{u}_3 &= \mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2. \end{aligned}$$

Show that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ if $i \neq j$.

Solution 2. Let us compute,

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_2 &= \mathbf{u}_1 \cdot (\mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1) \\ &= \mathbf{u}_1 \cdot \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 \cdot \mathbf{u}_1 \\ &= \mathbf{u}_1 \cdot \mathbf{v}_2 - \mathbf{u}_1 \cdot \mathbf{v}_2 \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_3 &= \mathbf{u}_1 \cdot (\mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2) \\ &= \mathbf{u}_1 \cdot \mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 \cdot \mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\underbrace{\mathbf{u}_1 \cdot \mathbf{u}_2}_{=0} \\ &= \mathbf{u}_1 \cdot \mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 \cdot \mathbf{u}_1 \\ &= \mathbf{u}_1 \cdot \mathbf{v}_3 - \mathbf{u}_1 \cdot \mathbf{v}_3 \\ &= 0. \end{aligned}$$

Next,

$$\begin{aligned}
\mathbf{u}_3 \cdot \mathbf{u}_2 &= (\mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2) \cdot \mathbf{u}_2 \\
&= \mathbf{v}_3 \cdot \mathbf{u}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1} \underbrace{\mathbf{u}_1 \cdot \mathbf{u}_2}_{=0} - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 \cdot \mathbf{u}_2 \\
&= \mathbf{v}_3 \cdot \mathbf{u}_2 - \mathbf{u}_2 \cdot \mathbf{v}_3 \\
&= 0.
\end{aligned}$$