VANDERBILT UNIVERSITY
MATH 208 — ORDINARY DIFFERENTIAL EQUATIONS
TEST 2.

NAME:

Question	Points
1 (20 pts)	
2 (20 pts)	
3 (20 pts)	
4 (20 pts)	
5 (20 pts)	
Extra Credit (5 pts)	
TOTAL (100 pts)	

Question 1 [20 pts]. Find the general solution of the systems below.

(a) $x' = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} x.$

Solution. Compute

$$\det \begin{bmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{bmatrix} = (2-\lambda)(-2-\lambda)+3 = \lambda^2 - 1 = 0,$$

giving $\lambda_1 = 1, \lambda_2 = -1$.

$\underline{\lambda_1 = 1}$. We solve

$$\begin{bmatrix} 2-\lambda_1 & -3 \\ 1 & -2-\lambda_1 \end{bmatrix} u = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} u = 0.$$

Writing $u = (z, w)$, we find $z = 3w$, so that eigenvectors are $u_1 = s(3, 1)$, where s is a free variable. A linearly independent solution is then $x_1 = e^t(3, 1)$.

$\underline{\lambda_2 = -1}$. We solve

$$\begin{bmatrix} 2-\lambda_2 & -3 \\ 1 & -2-\lambda_2 \end{bmatrix} u = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} u = 0.$$

Writing $u = (z, w)$, we find $z = w$, so that eigenvectors are $u_2 = s(1, 1)$, where s is a free variable. A second linearly independent solution is then $x_2 = e^{-t}(1, 1)$.

The general solution is $x = c_1 x_1 + c_2 x_2$.

(b) $x' = \begin{bmatrix} -1 & 2 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix} x.$

Hint: use your answer from (a).

Solution. Solutions x can be broken as $x = (X_1, X_2, 0, 0) + (0, 0, X_3, X_4)$, where (X_1, X_2) solves

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}' = \begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

and (X_3, X_4) solves

$$\begin{bmatrix} X_3 \\ X_4 \end{bmatrix}' = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} X_3 \\ X_4 \end{bmatrix}.$$

(X_3, X_4) was found in the previous question.

Compute

$$\det \begin{bmatrix} -1-\lambda & 2 \\ -1 & 3-\lambda \end{bmatrix} = (-1-\lambda)(3-\lambda) + 2 = 0,$$

giving $\lambda = 1 \pm \sqrt{2}$.

$\underline{\lambda = 1 + \sqrt{2}}$. We solve

$$\begin{bmatrix} -2-\sqrt{2} & 2 \\ -1 & 2-\sqrt{2} \end{bmatrix} u = 0.$$

Writing $u = (z, w)$, we find $z = (2-\sqrt{2})w$, so this gives $u = s(2-\sqrt{2}, 1)$, where s is a free variable.

$\underline{\lambda = 1 - \sqrt{2}}$. We solve

$$\begin{bmatrix} -2+\sqrt{2} & 2 \\ -1 & 2+\sqrt{2} \end{bmatrix} u = 0.$$

Writing $u = (z, w)$, we find $z = (2+\sqrt{2})w$, so this gives $u = s(2+\sqrt{2}, 1)$, where s is a free variable. We get that (X_1, X_2) is a linear combination of

$$e^{(1+\sqrt{2})t} \begin{bmatrix} 2-\sqrt{2} \\ 1 \end{bmatrix}$$

and

$$e^{(1-\sqrt{2})t} \begin{bmatrix} 2+\sqrt{2} \\ 1 \end{bmatrix}.$$

Hence

$$x_1 = e^{(1+\sqrt{2})t} \begin{bmatrix} 2-\sqrt{2} \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$x_2 = e^{(1-\sqrt{2})t} \begin{bmatrix} 2+\sqrt{2} \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$x_3 = e^t \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix},$$

and

$$x_3 = e^{-t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

are four linearly independent solutions, and $x = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ is the general solution.

Question 2 [20 pts]. Determine e^{At} if

$$A = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}.$$

Hint: You can use that

$$\begin{bmatrix} -4 & -2 & 5 \\ -5 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^{-1} = \frac{1}{25} \begin{bmatrix} 1 & -5 & 2 \\ -2 & 10 & 21 \\ 5 & 0 & 10 \end{bmatrix}.$$

Solution. Compute

$$\det \begin{bmatrix} 5-\lambda & -4 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & 5-\lambda \end{bmatrix} = -\lambda(\lambda-5)^2,$$

so $\lambda_1 = 0$ and $\lambda_2 = 5$ are the eigenvalues, with λ_2 of multiplicity two.

To find an eigenvector associated with λ_1, we solve

$$\begin{bmatrix} 5 & -4 & 0 & \vdots & 0 \\ 1 & 0 & 2 & \vdots & 0 \\ 0 & 2 & 5 & \vdots & 0 \end{bmatrix}.$$

Applying Gauss-Jordan elimination we find $u_1 = (-4, -5, 2)$, and $x_1 = e^{0t} u_1 = (-4, -5, 2)$ is a solution to $x' = Ax$.

Next, we move to λ_2, and consider:

$$\begin{bmatrix} 0 & -4 & 0 & \vdots & 0 \\ 1 & -5 & 2 & \vdots & 0 \\ 0 & 2 & 0 & \vdots & 0 \end{bmatrix}.$$

Applying Gauss-Jordan elimination, we find

$$\begin{bmatrix} 1 & 0 & 2 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Thus, this system has only one free variable, yielding only one linearly independent eigenvector which we can take to be $u_2 = (-2, 0, 1)$. Hence $x_2 = e^{5t}(-2, 0, 1)$ is a second linearly independent solution to $x' = Ax$. To find a third linearly independent solution, we need to find a generalized eigenvector associated with $\lambda_2 = 5$. Compute

$$(A - 5I)^2 = \begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix}^2 = \begin{bmatrix} -4 & 20 & -8 \\ -5 & 25 & -10 \\ 2 & -10 & 4 \end{bmatrix}.$$

Now we solve

$$\begin{bmatrix} -4 & 20 & -8 & \vdots & 0 \\ -5 & 25 & -10 & \vdots & 0 \\ 2 & -10 & 4 & \vdots & 0 \end{bmatrix}.$$

Applying Gauss-Jordan elimination gives

$$\left[\begin{array}{ccc:c} -1 & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which has two free variables that yield two linearly independent generalized eigenvectors $u_2 = (-2, 0, 1)$ and $u_3 = (5, 1, 0)$ (notice that we already knew from above that u_2 is a solution since it is an eigenvector). To find a third (linearly independent) solution to $x' = Ax$, compute

$$x_3 = e^{At}u_3 = e^{5t}(u_3 + t(A - 5I)u_3) = e^{5t}\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + te^{5t}\begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = e^{5t}\begin{bmatrix} 5-4t \\ 1 \\ 2t \end{bmatrix}.$$

A fundamental matrix is now given by $X = [x_1\, x_2\, x_3]$, i.e.,

$$X(t) = \begin{bmatrix} -4 & -2e^{5t} & e^{5t}(5-4t) \\ -5 & 0 & e^{5t} \\ 2 & e^{5t} & 2e^{5t}t \end{bmatrix}.$$

Recall that $e^{At} = X(t)(X(0))^{-1}$. Plugging $t = 0$ into $X(t)$ and using the hint we find

$$(X(0))^{-1} = \frac{1}{25}\begin{bmatrix} 1 & -5 & 2 \\ -2 & 10 & 21 \\ 5 & 0 & 10 \end{bmatrix}.$$

Thus,

$$e^{At} = X(t)(X(0))^{-1} = \frac{1}{25}\begin{bmatrix} -4 & -2e^{5t} & e^{5t}(5-4t) \\ -5 & 0 & e^{5t} \\ 2 & e^{5t} & 2e^{5t}t \end{bmatrix}\begin{bmatrix} 1 & -5 & 2 \\ -2 & 10 & 21 \\ 5 & 0 & 10 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} -4 + 29e^{5t} - 20te^{5t} & 20 - 20e^{5t} & -8 + 8e^{5t} - 40te^{5t} \\ -5 + 5e^{5t} & 25 & -10 + 10e^{5t} \\ 2 - 2e^{5t} + 10te^{5t} & -10 + 10e^{5t} & 4 + 21e^{5t} + 20te^{5t} \end{bmatrix}.$$

Question 3 [20 pts]. Find the general solution of

$$x' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 6e^{3t} \\ 2e^{3t} \end{bmatrix}.$$

Solution. There are two possible methods, one using undetermined coefficients and another using variation of parameters.

First, we compute the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. They are $\lambda_1 = 3$ and $\lambda_2 = -1$. Two eigenvectors associated with λ_1 and λ_2 are, respectively,

$$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

so that

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \text{ and } x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

are two linearly independent solutions of the associated homogeneous equation.

Solution using variation of parameters. From the previous calculations, we have a fundamental matrix

$$X(t) = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}.$$

Recalling that an invertible matrix of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has inverse given by

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

one immediately finds

$$(X(t))^{-1} = \frac{1}{2} \begin{bmatrix} e^{-3t} & e^{-3t} \\ -e^{t} & e^{t} \end{bmatrix}.$$

Next, invoke the formula

$$x_p = X(t) \int (X(t))^{-1} f(t)\, dt = \frac{1}{2} \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \int \begin{bmatrix} e^{-3t} & e^{-3t} \\ -e^{t} & e^{t} \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{3t}\, dt,$$

which gives

$$x_p = \frac{1}{2} \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \int \begin{bmatrix} 8 \\ -4e^{4t} \end{bmatrix} dt.$$

Performing the integral:

$$x_p = \begin{bmatrix} 4 \\ 4 \end{bmatrix} te^{3t} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}.$$

Solution using undetermined coefficients. As the inhomogeneous term in the equation is of the form (vector)$\times e^{3t}$, in order to find a particular solution, we try

$$x_p = ae^{3t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t}.$$

Plugging into the equation yields

$$3\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} = \begin{bmatrix} a_1 + 2a_2 \\ 2a_1 + a_2 \end{bmatrix} e^{3t} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{3t}.$$

This leads to

$$\begin{cases} 2a_1 - 2a_2 & = 6, \\ -2a_1 + 2a_2 & = 2, \end{cases}$$

which is an inconsistent system. Therefore, we change our initial guess and now attempt

$$x_p = ate^{3t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} te^{3t}.$$

Plugging into the equation,

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} = \begin{bmatrix} -2a_1 + 2a_2 \\ 2a_1 + 2a_2 \end{bmatrix} te^{3t} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{3t}.$$

Setting the terms with t and without t on each side equal to each other produces

$$\begin{cases} 2a_1 - 2a_2 & = 0, \\ -2a_1 + 2a_2 & = 0, \end{cases}$$

and

$$\begin{cases} a_1 & = 6, \\ a_2 & = 2. \end{cases}$$

It is impossible to satisfy both systems at the same time, thus, again, our attempt has failed to produce a particular solution.

Following the ideas developed in class, we now try

$$x_p = ate^{3t} + be^{3t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} te^{3t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^{3t}.$$

Plugging into the equation gives

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} 2a_1 - 2a_2 \\ -2a_1 + 2a_2 \end{bmatrix} te^{3t} + \begin{bmatrix} a_1 + 2b_1 - 2b_2 \\ a_2 - 2b_1 + 2b_2 \end{bmatrix} e^{3t}.$$

Setting the terms with t and without t on each side equal to each other,

$$\begin{cases} 2a_1 - 2a_2 & = 0, \\ -2a_1 + 2a_2 & = 0, \end{cases}$$

and

$$\begin{cases} a_1 + 2b_1 - 2b_2 & = 6, \\ a_2 - 2b_1 + 2b_2 & = 2. \end{cases}$$

In other words, we obtain the following system of four unknowns and four equations:

$$\begin{cases} 2a_1 - 2a_2 & = 0, \\ -2a_1 + 2a_2 & = 0, \\ a_1 + 2b_1 - 2b_2 & = 6, \\ a_2 - 2b_1 + 2b_2 & = 2. \end{cases}$$

Using Gauss-Jordan elimination, we find $a_1 = 4$, $a_2 = 4$, $b_1 = 1 + b_2$, and b_2 undetermined (i.e., a free variable). As discussed in class, we can set $b_2 = 0$, finally obtaining

$$x_p = \begin{bmatrix} 4 \\ 4 \end{bmatrix} te^{3t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t}.$$

The general solution is then $x = c_1 x_1 + c_2 x_2 + x_p$, where c_1 and c_2 are arbitrary constants.

<u>Remark</u>: To see that this agrees with the previous solution, write

$$\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

and recall that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$ is a solution of the associated homogeneous equation.

Question 4 [20 pts]. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be given by $F(x) = |x|^2 x$, where $|x|$ is the norm of x. What can you say about the existence and uniqueness of solutions of

$$\begin{cases} x' = F(x) \\ x(0) = x_0 \end{cases} \quad ?$$

Solution. We shall prove that the system has a unique solution defined on some time interval $(-\epsilon, \epsilon)$, $\epsilon > 0$, by showing that F is Lipschitz in a neighborhood of x_0. First notice that the map $x \mapsto |x|^2$ is the composition of a Lipschitz and smooth maps. Estimate:

$$\begin{aligned} |F(x) - F(y)| &= |\,|x|^2 x - |y|^2 y\,| \\ &= |\,|x|^2 x - |x|^2 y + |x|^2 y - |y|^2 y\,| \\ &\leq |\,|x|^2 x - |x|^2 y\,| + |\,|x|^2 y - |y|^2 y\,| \\ &= |x|^2\,|x - y| + ||x|^2 - |y|^2|\,|y| \\ &= |x|^2\,|x - y| + |y|\,|\,(|x| + |y|)(|x| - |y|)\,|, \\ &\leq |x|^2\,|x - y| + |y|(|x| + |y|)|x - y|, \end{aligned}$$

where in the last step we used that $|\,|x| - |y|\,| \leq |x - y|$. Let K be a constant such that $|x_0| < K$. Then, for all x, y such that $|x| \leq K$ and $|y| \leq K$, we have $|F(x) - F(y)| \leq 3K^2|x - y|$, and the result follows.

Question 5 [20 pts]. Let A be a (constant) $n \times n$ matrix, let $B(t)$ be an $n \times n$ matrix valued function, and let $f(t)$ a vector valued function. Suppose that $B(t)$ and $f(t)$ are continuous.

(a) Suppose that λ is an eigenvalue of A. Show that e^{λ} is an eigenvalue of e^A.

(b) Let $x_1, \ldots, x_n$ be n solutions of $x' = Ax$. Set

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}.$$

Prove that $X' = AX$.

(c) Let $Y(t)$ be a fundamental matrix for the system $x'(t) = B(t)x(t)$. Derive a formula for the function $v(t)$ so that $x_p(t) = Y(t)v(t)$ is a particular solution of the system $x'(t) = B(t)x(t) + f(t)$.

(d) What can you say about $e^{A+B(0)}$, where A and B are as above?

Solution.

(a) If $Ax = \lambda x$, $x \neq 0$, then $A^2x = A\lambda x = \lambda^2 x$, $A^3x = AA^2x = A\lambda^2 x = \lambda^3 x$, and so on. Thus

$$\begin{aligned} e^A x &= \sum_{n=0}^{\infty} \frac{A^n}{n!} x \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} x \\ &= e^{\lambda} x. \end{aligned}$$

(b) By the definition of multiplication of matrices, the j^{th} column of AX is given by Ax_j. But since x_j is a solution, i.e., $x_j' = Ax_j$, we have

$$\begin{aligned} AX &= \begin{bmatrix} Ax_1 & Ax_2 & \cdots & Ax_n \end{bmatrix} \\ &= \begin{bmatrix} x_1' & x_2' & \cdots & x_n' \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}' \\ &= X', \end{aligned}$$

as desired.

(c) Done in class.

(d) $e^{A+B(0)} \neq e^A e^B(0)$, unless A and $B(0)$ commute.

Extra credit [5 pts]. Let $A(t)$ be a $n \times n$ matrix valued function and $f(t)$ a vector valued function. Prove that the general solution of $x'(t) = A(t)x(t) + f(t)$ is of the form $x = x_h + x_p$, where x_h is a linear combination of n linearly independent solutions of the associated homogeneous system, and x_p is a particular solution.

Solution. Let y be any solution of the system. Since by hypothesis x_p is also a solution, the difference $y - x_p$ satisfies

$$(y - x_p)' = Ay + f - (Ax_p + f) = A(y - x_p),$$

i.e., $y - x_p$ satisfies the associated homogeneous equation. If $x_1, \ldots, x_n$ are n linearly independent solutions of $x' = Ax$, then $y - x_p$ can be written as a linear combination of $x_1, \ldots, x_n$. Thus, there exist constants $c_1, \ldots, c_n$ such that

$$y - x_p = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

or

$$y = c_1x_1 + c_2x_2 + \cdots + c_nx_n + x_p,$$

as desired.

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