VANDERBILT UNIVERSITY
MATH 208 — ORDINARY DIFFERENTIAL EQUATIONS
TEST 1.

NAME:

Question	Points
1 (5 pts)	
2 (10 pts)	
3 (15 pts)	
4 (15 pts)	
5 (15 pts)	
6 (20 pts)	
7 (5 pts)	
8 (5 pts)	
9 (10 pts)	
Extra Credit (5 pts)	
TOTAL (100 pts)	

Question 1 [5 pts]. For each equation below, identify the unknown function, classify the equation as linear or non-linear, and state its order.

(a) $y''' = 0$.
Unknown y, linear, third order.

(b) $\frac{d^2x}{dt^2} + x = \cos(t)$.
Unknown x, linear, second order.

(c) $\sqrt{y}y'' = x + 12$.
Unknown y, non-linear, second order.

(d) $e^{x^2} y \frac{dy}{dx} + xy = e^{-x}$.
Unknown y, non-linear, first order.

(e) $\cos x \frac{d^2y}{dx^2} + x^3 y = e^{-x}$.
Unknown y, linear, second order.

Question 2 [10 pts]. Solve the following initial value problems.

(a) $\sqrt{y}dx + (1+x)dy = 0$, $y(0) = 1$.

Separate variables to obtain

$$\int \frac{dy}{\sqrt{y}} = -\int \frac{dx}{1+x} \Rightarrow 2\sqrt{y} = -\ln|1+x| + C.$$

Plugging $x = 0$ and $y(0) = 1$, gives $2 = 0 + C$, thus

$$y = \frac{1}{4}(2 - \ln(1+x))^2, x > -1,$$

where we removed the absolute value for $x > -1$, since the initial condition is given at $x = 0$.

(b) $\dfrac{dy}{dx} = \dfrac{2y}{x} + (xy)^{-1}$, $y(1) = 3$.
Hint: use the substitution $u = y^2$.

Multiply the equation by y to get $yy' - \frac{2}{x}y^2 = \frac{1}{x}$. Let $u = y^2$, so that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2y\frac{dy}{dx} \Rightarrow y\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx}.$$

The equation becomes

$$\frac{du}{dx} - \frac{4}{x}u = \frac{2}{x},$$

Which is a linear equation for u with $p(x) = -\frac{4}{x}$ and $q(x) = \frac{2}{x}$. Compute:

$$\exp\int p(x)\,dx = \exp\int\left(-\frac{4}{x}\right)\,dx = x^{-4},$$

and

$$\int e^{\int p(x)\,dx} q(x)\,dx = \int \frac{2}{x^5}\,dx = -\frac{1}{2}x^{-4}.$$

Then

$$\begin{aligned} u(x) &= e^{-\int p(x)\,dx}\left(C + \int e^{\int p(x)\,dx} q(x)\,dx\right) \\ &= x^4\left(C - \frac{1}{2}x^{-4}\right) \\ &= Cx^4 - \frac{1}{2}. \end{aligned}$$

Changing back to y we find $y^2 = Cx^4 - \frac{1}{2}$. Using the initial condition: $y(1)^2 = 3^2 = -\frac{1}{2} + C \Rightarrow C = \frac{19}{2}$. thus

$$y = \sqrt{\frac{19x^4 - 1}{2}},$$

where we have chosen the positive square root in light of the initial condition.

Question 3 [15 pts]. Solve the following differential equations.

(a) $\dfrac{dx}{dt} - \dfrac{x}{t-1} = t^2 + 2.$

This is a linear equation with $p(t) = -\frac{1}{t-1}$ and $q(t) = t^2 + 2$. Compute:

$$e^{\int p(t)\,dt} = \exp \int \left(-\frac{1}{t-1}\right)\,dt = \frac{1}{t-1},\ t > 1,$$

and

$$\begin{aligned}\int e^{\int p(t)\,dt} q(t)\,dt &= \int \frac{t^2+2}{t-1}\,dt \\ &= \frac{1}{2}(-3 + 2t + t^2 + 6\ln|t-1|).\end{aligned}$$

Therefore,

$$\begin{aligned}x(t) &= e^{-\int p(t)\,dt}\left(C + \int e^{\int p(t)\,dt} q(t)\,dt\right) \\ &= \frac{1}{2}(-3 + 2t + t^2 + 6\ln(t-1)) + C\ln(t-1),\ t > 1.\end{aligned}$$

(b) $(x^2 - \dfrac{2}{y^3})y' = 3x^2 - 2xy.$

Write the equation as

$$(x^2 - \frac{2}{y^3})dy - (3x^2 - 2xy)\,dx = 0.$$

Letting $N(x,y) = x^2 - \frac{2}{y^3}$ and $M(x,y) = -3x^2 + 2xy$, we check that $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$. Thus, this is an exact equation. Set

$$F(x,y) = \int M(x,y)\,dx = \int (-3x^2 + 2xy)\,dx = x^2y - x^3 + g(y).$$

Next, compute

$$\frac{\partial F}{\partial y} = N \Rightarrow x^2 - 2y^{-3} = x^2 + g'(y) \Rightarrow g(y) = y^{-2}.$$

We conclude that the solution is given implicitly by

$$x^2y - x^3 + y^{-2} = C.$$

(c) $y + x\dfrac{dy}{dx} = \dfrac{1-y}{1+y}$.

Rewriting the equation as

$$x\frac{dy}{dx} = \frac{1-y}{1+y} - y = -\frac{y^2+2y-1}{y+1},$$

we see that this is a separable equation. Integrating:

$$\int \frac{y+1}{y^2+2y-1}\,dy = -\int \frac{dx}{x} \Rightarrow \frac{1}{2}\ln|y^2+2y-1| = -\ln|x| + C.$$

Exponentiating both sides yields

$$x^2(y^2+2y-1) = C.$$

Question 4 [15 pts]. Find the general solution of the given differential equation.
(a) $4y'' - 4y' + 10y = 0$.

The characteristic equation is

$$4\lambda^2 - 4\lambda + 10 = 0,$$

which has roots $\lambda = \frac{1 \pm 3i}{2}$. The general solution is given by

$$y = c_1 e^{\frac{t}{2}} \cos(\frac{3}{2}t) + c_2 e^{\frac{t}{2}} \sin(\frac{3}{2}t).$$

(b) $y'' - 2y' + y = 0$.

The characteristic equation is

$$\lambda^2 - 2\lambda + 1 = 0,$$

which has repeated roots $\lambda = 1$. The general solution is given by

$$y = c_1 e^t + c_2 t e^t.$$

(c) $t^2y'' + 7ty' - 7y = 0$, $t > 0$.

This is a Cauchy-Euler equations with $a = 1$, $b = 7$, and $c = -7$. The characteristic equation is

$$\lambda^2 + 6\lambda - 7 = 0,$$

which has roots $\lambda_1 = 1$ and $\lambda_2 = -7$. Thus

$$y = c_1 t + c_2 t^{-7}.$$

(d) $t^2y'' + 7ty' - 7y = 0$, $t < 0$.

This is the same equation as in (c), but now for negative values of t. Letting $\tau = -t$ and using

$$\frac{dy}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt} = -\frac{dy}{d\tau},$$

and

$$\frac{d^2y}{dt^2} = -\frac{d}{dt}\frac{dy}{d\tau} = \frac{d^2y}{d\tau^2}.$$

The equation becomes

$$\tau^2\frac{d^2y}{d\tau^2} + 7\tau\frac{dy}{d\tau} - 7y = 0, \tau > 0,$$

which has solution

$$y = c_1\tau + c_2\tau^{-7}.$$

In terms of $t < 0$:

$$y = -c_1 t - c_2 t^{-7}.$$

Question 5 [15 pts]. Give the form of the particular solution for the given differential equations. You do not have to find the values of the constants of the particular solution.

(a) $y'' - y' - 12y = t^6 e^{-3t}$.

The characteristic equation is

$$\lambda^2 - \lambda - 12 = 0,$$

which has roots $\lambda_1 = 4$ and $\lambda_2 = -3$. Hence, the solutions of the homogeneous equation are $y_1 = e^{4t}$ and $y_2 = e^{-3t}$. Since the term e^{-3t} repeats a solution of the homogeneous equation, we take

$$y_p = (a_6 t^t + a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0) t e^{-3t}.$$

(b) $y'' - 2y' - 3y = t^2 + 7$.

The characteristic equation is

$$\lambda^2 - 2\lambda - 3 = 0,$$

which has roots $\lambda_1 = -1$ and $\lambda_2 = 3$. Hence, the solutions of the homogeneous equation are $y_1 = e^{-t}$ and $y_2 = e^{3t}$, and we take

$$y_p = a_2 t^2 + a_1 t + a_0.$$

(c) $y'' + 6y' + 9y = te^{-3t}$.

The characteristic equation is

$$\lambda^2 + 6\lambda + 9 = 0,$$

which has repeated roots $\lambda = -3$. Hence, the solutions of the homogeneous equation are $y_1 = e^{-3t}$ and $y_2 = te^{-3t}$, and we take

$$y_p = (At + B)t^2 e^{-3t}.$$

(d) $y'' - y' - 12y = t^6 \cos(-3t)$.

The characteristic equation is

$$\lambda^2 - \lambda - 12 = 0,$$

which has roots $\lambda_1 = 4$ and $\lambda_2 = -3$. Hence, the solutions of the homogeneous equation are $y_1 = e^{4t}$ and $y_2 = e^{-3t}$, and we take

$$\begin{aligned} y_p = &(a_6t^t + a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0)\cos(-3t) \\ &+(b_6t^t + b_5t^5 + b_4t^4 + b_3t^3 + b_2t^2 + b_1t + b_0)\sin(-3t). \end{aligned}$$

Question 6 [20 pts]. Consider the differential equation:

$$(1-x)y'' + xy' - y = \sin x,\ 0 < x < 1,$$

Knowing that $y_1 = e^x$ is a solution of the associate homogeneous problem, find a particular solution solving the non-homogeneous problem.

Another linearly independent solution to the homogeneous equation is given by

$$\begin{aligned} y_2(x) &= y_1(x)\int \frac{e^{-\int p(x)\,dx}}{y_1(x)^2}\,dx = e^x \int \frac{e^{-\int \frac{x}{1-x}\,dx}}{e^{2x}}\,dx = e^x \int \frac{e^{\int (1-\frac{1}{1-x})\,dx}}{e^{2x}}\,dx \\ &= e^x \int \frac{e^{x+\ln(1-x)}}{e^{2x}}\,dx = e^x \int e^{-x}(1-x)\,dx = e^x e^{-x} x = x. \end{aligned}$$

Variation of parameters now gives

$$\begin{aligned} y_p(x) &= -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\,dx \\ &= -e^x(x)\int \frac{x\sin x}{e^x(1-x)^2}\,dx + x\int \frac{\sin x}{(1-x)^2}\,dx, \end{aligned}$$

where we used $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = e^x(1-x)$.

Question 7 [5 pts]. Show that the problem

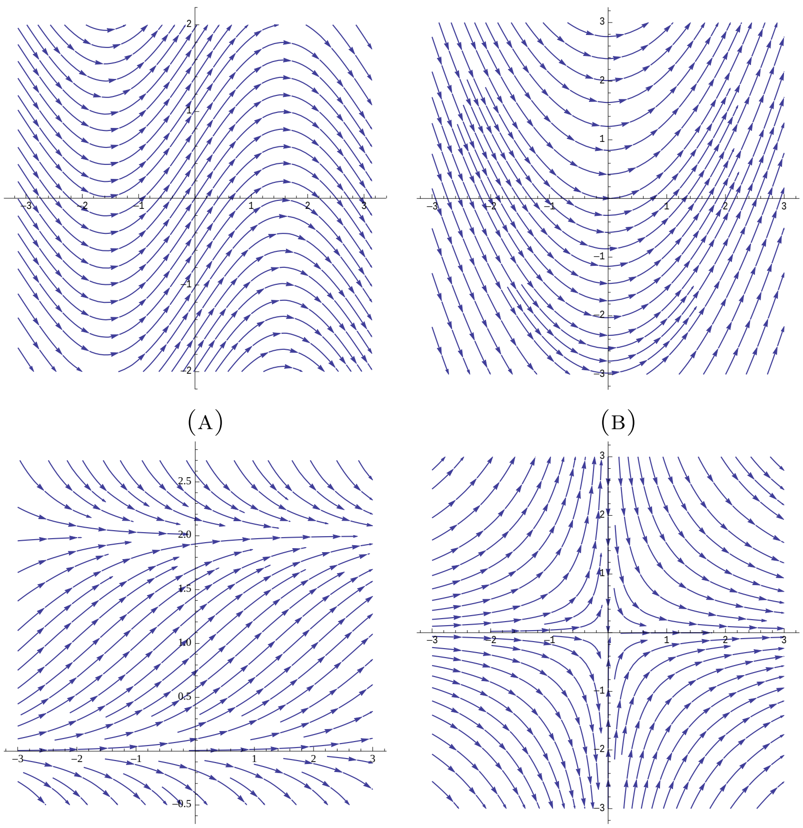
$$x' = 3x^{\frac{2}{3}},\ x(0) = 1.$$

has a unique solution defined in some neighborhood of $t = 0$.

Let $f(t,x) = 3x^{\frac{2}{3}}$, so that $\partial_x f(t,x) = 2x^{-\frac{1}{3}}$. Since both f and $\partial_x f$ are continuous in the neighborhood of $(0,1)$, the result follows from the existence and uniqueness theorem for first order equations.

Question 8 [5 pts]. Match the direction fields with the given differential equations.

(a) $y' = -\dfrac{y}{x}$ (b) $y' = \cos x$ (c) $y' = y(1 - 0.5y)$ (d) $y' = x$



(A) (B) (C) (D)

(a) = D, (b) = A, (c) = C, (d) = B

Question 9 [10 pts]. Consider the initial value problem:

$$\begin{cases} y' = y, \\ y(0) = 1. \end{cases}$$

Show that the n^{th} approximation y_n in the Euler method, with a step size $\frac{1}{n}$, is given by

$$y_n = \left(1 + \frac{1}{n}\right)^n.$$

We start with $x_0 = 0$, and $y_0 = 1$. Then

$$\begin{aligned} y_n &= y_{n-1} + \frac{1}{n} y_{n-1} = y_{n-1}\left(1 + \frac{1}{n}\right) = \left[y_{n-2}\left(1 + \frac{1}{n}\right)\right]\left(1 + \frac{1}{n}\right) \\ &= y_{n-2}\left(1 + \frac{1}{n}\right)^2 = \cdots = y_0\left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n. \end{aligned}$$

Extra credit [5 pts]. Let h be a positive function, and f a given function. Show that there exists at most one solution to the differential equation

$$y''(t) - h(t)y(t) = f(t),\ 0 \leq t \leq 1,$$

satisfying $y(0) = y(1) = 0$.

It suffices to show that $y \equiv 0$ is the only solution when $f(t) = 0$. Multiply the equation by $-y$, integrate between 0 and 1, and integrate by parts to get

$$\int_0^1 (y'(t))^2 \, dt + \int_0^1 h(t)(y(t))^2 \, dt = 0.$$

Since both terms on the left hand side are non-negative and $h(t) > 0$, one must have $y(t) = 0$.

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