VANDERBILT UNIVERSITY
MATH 208 — ORDINARY DIFFERENTIAL EQUATIONS
PRACTICE TEST 2.

Question 1. Let

$$A = \begin{bmatrix} t & \sin t \\ \cos t & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & e^t \\ \sin t & 2 \end{bmatrix}.$$

Compute AB, BA, $\frac{d}{dt}A$, $\frac{d}{dt}B$, and $\frac{d}{dt}(AB)$.

Solution.

$$AB = \begin{bmatrix} t + \sin^2 t & te^t + 2\sin t \\ \cos t + \sin t & 2 + e^t \cos t \end{bmatrix}, \; BA = \begin{bmatrix} t + e^t \cos t & e^t + \sin t \\ 2\cos t + t \sin t & 2 + \sin^2 t \end{bmatrix}.$$

$$A' = \begin{bmatrix} 1 & \cos t \\ -\sin t & 0 \end{bmatrix}, \; B' = \begin{bmatrix} 0 & e^t \\ \cos t & 0 \end{bmatrix},$$

$$(AB)' = \begin{bmatrix} 1 + 2\cos t \sin t & e^t + te^t + 2\cos t \\ \cos t - \sin t & e^t \cos t - e^t \sin t \end{bmatrix}.$$

Question 2. Give an example of two matrices such that $AB \neq BA$.

Solution. Take

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & -3 \\ 3 & -5 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 5 & -13 \\ 9 & -29 \end{bmatrix}, \; BA = \begin{bmatrix} -10 & -14 \\ -12 & -14 \end{bmatrix}.$$

Question 3. Let $A(t)$ be a $n \times n$ matrix valued function and $f(t)$ a vector valued function. Prove that the general solution of $x'(t) = A(t)x(t) + f(t)$ is of the form $x = x_h + x_p$, where x_h is a linear combination of n linearly independent solutions of the associated homogeneous system, and x_p is a particular solution.

Solution. Let y be any solution of the system. Since by hypothesis x_p is also a solution, the difference $y - x_p$ satisfies

$$(y - x_p)' = Ay + f - (Ax_p + f) = A(y - x_p),$$

i.e., $y - x_p$ satisfies the associated homogeneous equation. If $x_1, \dots, x_n$ are n linearly independent solutions of $x' = Ax$, then $y - x_p$ can be written as a linear combination of $x_1, \dots, x_n$. Thus, there exist constants $c_1, \dots, c_n$ such that

$$y - x_p = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n,$$

or

$$y = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n + x_p,$$

as desired.

Question 4. Let A be a constant $n \times n$ matrix and let $x_1, \ldots, x_n$ be n linearly independent solutions of $x' = Ax$. Set

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}.$$

Prove that $X' = AX$.

<u>Solution.</u> By the definition of multiplication of matrices, the j^{th} column of AX is given by Ax_j. But since x_j is a solution, i.e., $x_j' = Ax_j$, we have

$$\begin{aligned} AX &= \begin{bmatrix} Ax_1 & Ax_2 & \cdots & Ax_n \end{bmatrix} \\ &= \begin{bmatrix} x_1' & x_2' & \cdots & x_n' \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}' \\ &= X', \end{aligned}$$

as desired. Notice that the assumption that $x_1, \ldots, x_n$ are linearly independent is not necessary.

Question 5. Let A be a real $n \times n$ symmetric matrix. Prove that all eigenvalues of A are real.

<u>Solution.</u> Let λ be an eigenvalue with associated eigenvector x:

$$Ax = \lambda x. \tag{1}$$

Take the complex conjugate of (1) to obtain

$$A\overline{x} = \overline{\lambda}\overline{x}, \tag{2}$$

where $^{-}$ denotes the complex conjugate and we used that $\overline{A} = A$ since A is real. Multiply (1) on the left by $\overline{x}^T$, (2) on the left by x^T, where ^{T} denotes the transpose, and subtract to get

$$\overline{x}^T Ax - x^T A\overline{x} = \lambda \overline{x}^T x - \overline{\lambda} x^T \overline{x}. \tag{3}$$

Write $x = (x_1, \ldots, x_n)$, denote the ij^{th} entry of A by a_{ij}, and compute

$$\begin{aligned} \overline{x}^T Ax &= [\overline{x}_1\, \overline{x}_2 \cdots \overline{x}_n] \begin{bmatrix} \sum_{j=1}^n a_{1j}x_j \\ \sum_{j=1}^n a_{2j}x_j \\ \vdots \\ \sum_{j=1}^n a_{nj}x_j \end{bmatrix} \\ &= \overline{x}_1 \sum_{j=1}^n a_{1j}x_j + \overline{x}_2 \sum_{j=1}^n a_{2j}x_j + \cdots + \overline{x}_n \sum_{j=1}^n a_{nj}x_j \\ &= \sum_{i=1}^n \overline{x}_i \sum_{j=1}^n a_{ij}x_j \\ &= \sum_{i=1}^n \sum_{j=1}^n a_{ij}\overline{x}_i x_j \\ &= \sum_{j=1}^n \sum_{i=1}^n a_{ij}\overline{x}_i x_j \\ &= \sum_{j=1}^n x_j \sum_{i=1}^n a_{ij}\overline{x}_i \end{aligned}$$

$$
\begin{aligned}
&= x_1 \sum_{i=1}^n a_{i1}\overline{x}_i + x_2 \sum_{i=1}^n a_{i2}\overline{x}_i + \cdots + x_n \sum_{i=1}^n a_{in}\overline{x}_i \\
&= [x_1\, x_2 \cdots x_n] \begin{bmatrix} \sum_{i=1}^n a_{i1}\overline{x}_i \\ \sum_{i=1}^n a_{i2}\overline{x}_i \\ \vdots \\ \sum_{i=1}^n a_{in}\overline{x}_i \end{bmatrix} \\
&= x^T A^T \overline{x} \\
&= x^T A \overline{x},
\end{aligned}
$$

where in the last step we used that $A^T = A$ since A is symmetric by assumption. Summarizing, $\overline{x}^T Ax = x^T A\overline{x}$, and therefore the left hand side of (3) vanishes. Next, observe that

$$
\begin{aligned}
\overline{x}^T x &= [\overline{x}_1\, \overline{x}_2 \cdots \overline{x}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= \overline{x}_1 x_1 + \overline{x}_2 x_2 + \cdots + \overline{x}_n x_n \\
&= [x_1\, x_2 \cdots x_n] \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_n \end{bmatrix} \\
&= x^T \overline{x}.
\end{aligned}
$$

Therefore, (3) gives

$$
(\lambda - \overline{\lambda})\overline{x}^T x = 0. \tag{4}
$$

Recall that for any complex number z, $\overline{z}z$ is real an in fact $\overline{z}z \geq 0$, with equality if and only if $z = 0$. As $\overline{x}^T x = \overline{x}_1 x_1 + \overline{x}_2 x_2 + \cdots + \overline{x}_n x_n$ and x is not zero because it is an eigenvector, (4) implies $\lambda = \overline{\lambda}$, as desired.

Question 6. Find a general solution of $x' = Ax$ for the given matrices A:

(a) $\begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}$.

Solution. Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = 3$. Eigenvectors $(3, 2)$, $(1, 1)$. Solutions: $x_1 = e^{4t}(3, 2)$, $x_2 = e^{3t}(1, 1)$.

(b) $\begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}$.

Solution. Eigenvalues: $\lambda = 2 \pm 3i$. Eigenvectors $(-1 \pm 3i, 5) = (-1, 5) \pm i(3, 0)$. Solutions: $x_1 = e^{2t}\cos(3t)(-1, 5) - e^{2t}\sin(3t)(3, 0)$, $x_2 = e^{2t}\sin(3t)(-1, 5) + e^{2t}\cos(3t)(3, 0)$.

(c) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$.

Solution. Notice that is suffices to consider the matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Eigenvalues: $\lambda_1 = \lambda_2 = 3$, $\lambda_3 = \lambda_4 = -1$. Eigenvectors $(0,0,1,1)$, $(1,1,0,0)$, $(0,0,-1,1)$, $(-1,1,0,0)$. Solutions: $x_1 = e^{3t}(0,0,1,1)$, $x_2 = e^{3t}(1,1,0,0)$, $x_3 = e^{-t}(0,0,-1,1)$ $e^{-t}(-1,1,0,0)$.

(d) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution. Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 1$. Eigenvectors $(0,0,1)$, and $(1,0,0)$. Thus, there is only one linearly independent eigenvector associated with the eigenvalue 1. Two linearly independent solutions are $x_1 = e^{2t}(0,0,1)$ and $x_2 = e^{t}(1,0,0)$. Computing $(A-1I)^2$ we find

$$(A-1I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Two linearly independent solutions to $(A-1I)^2u = 0$ are $(1,0,0)$ and $(0,1,0)$. We finally find

$$\begin{aligned} x_3 = e^{At}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &= e^{t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + e^{t}t(A-1I)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= e^{t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + e^{t}t\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e^{t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}. \end{aligned}$$

Question 7. Find a general solution of $x' = Ax + f$ for the given A and f:

(a) $\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 1 \\ 9 & 3 & -4 \end{bmatrix}$, $f(t) = \begin{bmatrix} t \\ 0 \\ 1 \end{bmatrix}$.

Solution.

$$x = c_1e^{-t}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + c_2e^{-t}\begin{bmatrix} t \\ 1 \\ 3t \end{bmatrix} + c_3e^{-t}\begin{bmatrix} -t+\frac{1}{2}t^2 \\ t \\ 1-3t+\frac{3}{2}t^2 \end{bmatrix} + \begin{bmatrix} 2+t \\ 7-3t \\ 10 \end{bmatrix}.$$

(b) $\begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & 2 \\ 0 & -1 & 2 \end{bmatrix}$, $f(t) = \begin{bmatrix} e^{-t} \\ 2 \\ 1 \end{bmatrix}$.

Solution.

$$x = c_1e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2e^{\frac{5}{2}t}\cos\frac{\sqrt{7}t}{2}\begin{bmatrix} 11 \\ 2 \\ 4 \end{bmatrix} - c_2e^{\frac{5}{2}t}\sin\frac{\sqrt{7}t}{2}\begin{bmatrix} -3\sqrt{7} \\ -2\sqrt{7} \\ 0 \end{bmatrix}$$

$$+ c_3 e^{\frac{5}{2}t} \sin \frac{\sqrt{7}t}{2} \begin{bmatrix} 11 \\ 2 \\ 4 \end{bmatrix} + c_3 e^{\frac{5}{2}t} \cos \frac{\sqrt{7}t}{2} \begin{bmatrix} -3\sqrt{7} \\ -2\sqrt{7} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3}e^{-t} + \frac{11}{16} \\ -\frac{1}{4} \\ -\frac{5}{8} \end{bmatrix}.$$

Question 8. Show that in general it is not true that $e^{A+B} = e^A e^B$, where A and B are $n \times n$ matrices.

Solution. Take

$$X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, Y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$e^X = \begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix}, e^Y = \begin{bmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{bmatrix}, e^{X+Y} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

Indeed, notice that if

$$A = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix},$$

then

$$A^2 = \begin{bmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{bmatrix},$$
$$A^3 = \begin{bmatrix} 0 & \theta^3 \\ -\theta^3 & 0 \end{bmatrix},$$
$$A^4 = \begin{bmatrix} \theta^4 & 0 \\ 0 & \theta^4 \end{bmatrix},$$
$$A^5 = \begin{bmatrix} 0 & -\theta^5 \\ \theta^5 & 0 \end{bmatrix},$$

and so on. Thus

$$\begin{aligned} e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{bmatrix} \\ &+ \frac{1}{3!} \begin{bmatrix} 0 & \theta^3 \\ -\theta^3 & 0 \end{bmatrix} + \frac{1}{4!} \begin{bmatrix} \theta^4 & 0 \\ 0 & \theta^4 \end{bmatrix} + \frac{1}{5!} \begin{bmatrix} 0 & -\theta^5 \\ \theta^5 & 0 \end{bmatrix} + \cdots \\ &= \begin{bmatrix} 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots & -\theta + \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \cdots \\ \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots & 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \end{aligned}$$

On the other hand,

$$X + Y = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},$$

and

$$(X+Y)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which gives the result.
Note: If you are curious about a formula for e^{A+B}, google "Baker-Campbell-Hausdorff formula."

Question 9. Find e^{At} if

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 5 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

<u>Solution.</u> The eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = \lambda_3 = 3$. We readily find that $(2, 1, 0)$ is an eigenvector associated with λ_1, thus $x_1 = e^{5t}(2, 1, 0)$ is a solution. For $\lambda_2 = \lambda_3 = 3$, one finds only one linearly independent eigenvector, namely, $(1, 0, 0)$, which gives $x_2 = e^{3t}(1, 0, 0)$. Next, we seek the generalized eigenvectors. Computing,

$$(A - 3I)^2 = \begin{bmatrix} 0 & 8 & 16 \\ 0 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solving $(A - 3I)^2 u = 0$ yields $u_2 = (1, 0, 0)$ (which we already knew) and $u_3 = (0, 2, 1)$. Therefore:

$$x_3 = e^{3t}(u_3 + t(A - 3I)u_3) = e^{3t}(3t, 2, -1).$$

A fundamental matrix is now given by

$$X(t) = \begin{bmatrix} 2e^{5t} & e^{3t} & 3te^{3t} \\ e^{5t} & 0 & 2e^{3t} \\ 0 & 0 & -e^{3t} \end{bmatrix}.$$

From this we find

$$(X(0))^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -2 & -4 \\ 0 & 0 & -1 \end{bmatrix},$$

and thus

$$e^{At} = X(t)(X(0))^{-1} = \begin{bmatrix} e^{3t} & 2e^{5t} - 2e^{3t} & 4e^{5t} - (4 + 3t)e^{3t} \\ 0 & e^{5t} & 2e^{5t} - 2e^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix}.$$

Question 10. Let A be a square matrix and suppose that λ is an eigenvalue of A.

(a) Show that e^{λ} is an eigenvalue of e^{A}.

(b) Show that if B is an invertible matrix, then $B^{-1}e^{A}B = e^{B^{-1}AB}$.

<u>Solution.</u> If $Ax = \lambda x$, $x \neq 0$, then $A^2x = A\lambda x = \lambda^2 x$, $A^3x = AA^2x = A\lambda^2 x = \lambda^3 x$, and so on. Thus

$$\begin{aligned} e^{A}x &= \sum_{n=0}^{\infty} \frac{A^n}{n!}x \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}x \\ &= e^{\lambda}x. \end{aligned}$$

For part (b), notice that

$$\begin{aligned} (BAB^{-1})^2 &= (BAB^{-1})(BAB^{-1}) = BA^2B^{-1}, \\ (BAB^{-1})^3 &= (BAB^{-1})(BAB^{-1})(BAB^{-1}) = BA^3B^{-1}, \\ &\vdots \\ (BAB^{-1})^n &= (BAB^{-1})(BAB^{-1})\cdots(BAB^{-1}) = BA^nB^{-1}, \end{aligned}$$

from which the result immediately follows.

Question 11. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be given by $F(x) = |x|x$, where $|x|$ is the norm of x. What can you say about the existence and uniqueness of solutions of

$$\begin{cases} x' = F(x) \\ x(0) = x_0 \end{cases} ?$$

Solution. We shall prove that the system has a unique solution defined on some time interval $(-\epsilon, \epsilon)$, $\epsilon > 0$, by showing that F is Lipschitz in a neighborhood of x_0.

$$\begin{aligned} |F(x) - F(y)| &= |\,|x|x - |y|y\,| \\ &= |\,|x|x - |x|y + |x|y - |y|y\,| \\ &\leq |\,|x|x - |x|y\,| + |\,|x|y - |y|y\,| \\ &= |x|\,|x-y| + ||x| - |y||\,|y| \\ &\leq |x|\,|x-y| + |y|\,|x-y|, \end{aligned}$$

where in the last step we used that $|\,|x| - |y|\,| \leq |x-y|$. Let K be a constant such that $|x_0| < K$. Then, for all x, y such that $|x| \leq K$ and $|y| \leq K$, we have $|F(x) - F(y)| \leq 2K|x-y|$, and the result follows.

Question 12. Prove the several statements that were left as exercise in class. In other words, many of the properties/statements studied in chapter 9 have not been proven in class, but rather I indicated that I would leave them as an exercise; do those.

URL: `http://www.disconzi.net/Teaching/MAT208-Fall-14/MAT208-Fall-14.html`