VANDERBILT UNIVERSITY
MATH 208 — ORDINARY DIFFERENTIAL EQUATIONS
PRACTICE TEST 2.

Question 1. Let

$$A = \begin{bmatrix} t & \sin t \\ \cos t & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & e^t \\ \sin t & 2 \end{bmatrix}.$$

Compute AB, BA, $\frac{d}{dt}A$, $\frac{d}{dt}B$, and $\frac{d}{dt}(AB)$.

Question 2. Give an example of two matrices such that $AB \neq BA$.

Question 3. Let $A(t)$ be a $n \times n$ matrix valued function and $f(t)$ a vector valued function. Prove that the general solution of $x'(t) = A(t)x(t) + f(t)$ is of the form $x = x_h + x_p$, where x_h is a linear combination of n linearly independent solutions of the associated homogeneous system, and x_p is a particular solution.

Question 4. Let A be a constant $n \times n$ matrix and let $x_1, \dots, x_n$ be n linearly independent solutions of $x' = Ax$. Set

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}.$$

Prove that $X' = AX$.

Question 5. Let A be a real $n \times n$ symmetric matrix. Prove that all eigenvalues of A are real.

Question 6. Find a general solution of $x' = Ax$ for the given matrices A:

(a) $\begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Question 7. Find a general solution of $x' = Ax + f$ for the given A and f:

(a) $\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 1 \\ 9 & 3 & -4 \end{bmatrix}$, $f(t) = \begin{bmatrix} t \\ 0 \\ 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & 2 \\ 0 & -1 & 2 \end{bmatrix}$, $f(t) = \begin{bmatrix} e^{-t} \\ 2 \\ 1 \end{bmatrix}$.

Question 8. Show that in general it is not true that $e^{A+B} = e^A e^B$, where A and B are $n \times n$ matrices. Note: If you are curious about a formula for e^{A+B}, google "Baker-Campbell-Hausdorff formula."

Question 9. Find e^{At} if

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 5 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

Question 10. Let A be a square matrix and suppose that λ is an eigenvalue of A.

(a) Show that e^λ is an eigenvalue of e^A.

(b) Show that if B is an invertible matrix, then $B^{-1}e^A B = e^{B^{-1}AB}$.

Question 11. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be given by $F(x) = |x|x$, where $|x|$ is the norm of x. What can you say about the existence and uniqueness of solutions of

$$\begin{cases} x' = F(x) \\ x(0) = x_0 \end{cases} \quad ?$$

Question 12. Prove the several statements that were left as exercise in class. In other words, many of the properties/statements studied in chapter 9 have not been proven in class, but rather I indicated that I would leave them as an exercise; do those.

URL: `http://www.disconzi.net/Teaching/MAT208-Fall-14/MAT208-Fall-14.html`