VANDERBILT UNIVERSITY
MATH 208 — ORDINARY DIFFERENTIAL EQUATIONS
PRACTICE FINAL.

The following directions must be followed in the Final Exam. Keep them in mind while preparing for the test.

Instructions. Show all work on your exam paper. Little or no credit may be awarded—even if the final answer is correct—if a full explanation is not provided. Write legibly. Make statements (for instance, write equal signs if you think two expressions are equal.)

Question 1. Find the general solution of the differential equations below.

(a) $x'' - x' - 6x = 0$.

Solution. The characteristic roots are $\lambda = -2$ and $\lambda = 3$, thus

$$x = c_1 e^{-2t} + c_2 e^{3t}.$$

(b) $x'' + 8x' + 16x = 0$.

Solution. The characteristic roots are $\lambda = -4$ with multiplicity two, thus

$$x = c_1 e^{-4t} + c_2 t e^{-4t}.$$

Question 2. Consider the initial value problem.

$$\frac{dx}{dt} + t\cos x = 0,\ x(1) = \pi. \tag{1}$$

(a) What does the existence and uniqueness theorems for differential equations say about solvability of (1)? Notice that we have learned more than one existence and uniqueness theorem. Apply whichever one you find appropriate and, in doing so,

- State and check the assumptions of the theorem you are using.
- Make precise statements, and state where the solution is defined.

Solution. Use the following

Theorem 1. *(page 11 of the textbook) Consider the initial value problem $x' = f(t,x)$, $x(t_0) = x_0$. If f and $\partial_x f$ are continuous in some rectangle $R = \{(t,x) : a < t < b,\ c < x < d\}$ that contains the point (t_0, x_0), then the initial value problem has a unique solution in some interval $t_0 - \delta < t < t_0 + \delta$, where δ is a positive number.*

Write $x' = f(t,x)$, with $f(t,x) = -t\cos x$, $\partial_x f(t,x) = t\sin x$, and $(t_0, x_0) = (1, \pi)$. f and $\partial_x f$ are everywhere continuous. Hence, there exists a solution defined on an open interval containing $t_0 = 1$.

(b) Solve the initial value problem (1).

Solution. Write

$$\frac{dx}{\cos x} = -t\,dt.$$

Integrating

$$\ln|\sec x + \tan x| = -\frac{1}{2}t^2 + C.$$

Plugging $t = 1$, $x = \pi$,

$$\ln|\sec\pi + \tan\pi| = \ln|-1| = 0 = -\frac{1}{2} + C \Rightarrow C = \frac{1}{2}.$$

Therefore, the solution is given implicitly by

$$\ln|\sec x + \tan x| = -\frac{1}{2}t^2 + \frac{1}{2}.$$

Question 3. Given an $n \times n$ matrix A:

(a) What is the definition of a fundamental matrix for $x' = Ax$? If $X(t)$ is a fundamental matrix, what is its relation to e^{At}?

Solution. A fundamental matrix $X(t)$ for $x' = Ax$ is a matrix whose columns are linearly independent solutions of $x' = Ax$ (page 521 of the textbook). It can be related to e^{At} by (page 552 of the textbook).

$$e^{At} = X(t)(X(0))^{-1}.$$

(b) What is the definition of a generalized eigenvector of A?

Solution. A generalized eigenvector is a non-zero vector $u \in \mathbb{R}^n$ satisfying

$$(A - \lambda I)^m u = 0$$

for some scalar λ and some positive integer m, where I is the identity matrix (page 553 of the textbook).

(c) Find a formula for the general solution of the system

$$x'(t) = Ax(t) + f(t).$$

Solution. The general solution can be written as $x = x_h + x_p$, where x_h is a linear combination of n linearly independent solutions of the associated homogeneous equation. We seek a formula for x_p. Let $X(t)$ be a fundamental matrix for the homogeneous system, and set $x_p = Xv$. Plugging into the equation and using that $X' = AX$, we find

$$Xv' = f,$$

so that

$$x_p(t) = X(t) \int (X(t))^{-1} f(t)\, dt.$$

Details can be found in the proof given in class or on pages 544-546 of the textbook.

Question 4. Consider the matrix

$$A = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}.$$

The characteristic polynomial of A is given by $p(\lambda) = -\lambda(\lambda - 5)^2$ and

$$(A - 5I)^2 = \begin{bmatrix} -4 & 20 & -8 \\ -5 & 25 & -10 \\ 2 & -10 & 4 \end{bmatrix}.$$

Using generalized eigenvectors, find a fundamental matrix $X(t)$ for the system

$$x' = Ax.$$

Solution. $\lambda = 0$ is an eigenvalue with multiplicity one, and $\lambda = 5$ is an eigenvalue with multiplicity two. First we solve

$$(A - 0I)u_1 = 0,$$

and find, via an application of Gauss-Jordan elimination, that $u = (-4, -5, 2)$ (up to a multiplicative constant). Next, we solve

$$(A - 5I)u_2 = 0,$$

and find one linearly independent eigenvector given by $u_2 = (-2, 0, 1)$ (up to a multiplicative constant). Since $\lambda = 2$ has multiplicity two, we seek a generalized eigenvector satisfying

$$(A - 5I)^2 u = 0.$$

We find $u = su_2 + ru_3$, where s and r are arbitrary constants, u_2 is as above and $u_3 = (5, 1, 0)$. Three linearly independent solutions are given by $x_1 = e^{0t}u_1 = (-4, -5, 2)$, $x_2 = e^{5t}u_2 = (-2e^{5t}, 0, e^{5t})$, and

$$x_3 = e^{At}u_3 = e^{5t}(u_3 + t(A - 5I)u_3) = (5e^{5t} - 4te^{5t}, e^{5t}, 2te^{5t}).$$

Then $X(t) = [x_1\ x_2\ x_3]$.

Question 5. Consider the system

$$\begin{cases} x' = 2x + y + 9, \\ y' = -5x - 2y - 22. \end{cases}$$

(a) Determine the nature (saddle point, spiral, node, etc) and stability (stable, asymptotic stable, etc) of the critical points.

Solution. First we find the critical points solving

$$\begin{cases} 2x + y + 9 = 0, \\ -5x - 2y - 22 = 0. \end{cases}$$

We obtain $x = -4$, $y = -1$. The eigenvalue techniques we learned apply to a critical point that is the origin, thus we need to make a change of variables. Set $u = x + 4$ and $v = y + 1$. The system becomes

$$\begin{cases} u' = 2u + v, \\ v' = -5u - 2v. \end{cases}$$

The eigenvalues are $\lambda + \pm i$, and therefore $(0, 0)$ is a stable center for the u, v system, and we conclude that $(-4, -1)$ is a stable center for the original system.

(b) Compute the eigenvectors.

Solution. Gauss-Jordan yields $(-2 \pm i, 5)$.

(c) Sketch the phase plane diagram.

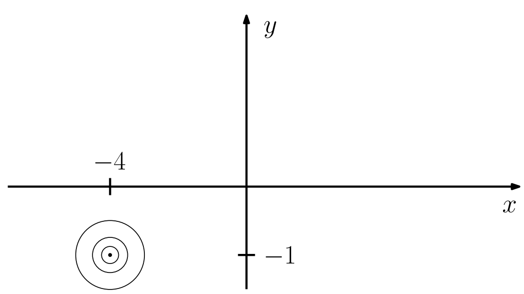

FIGURE 1. Phase diagram rough sketch.

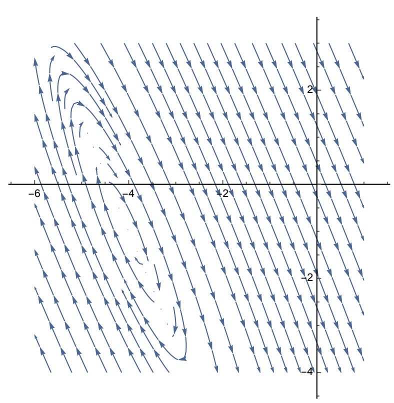

FIGURE 2. Computer generated phase diagram.

Question 6. Consider the system

$$\begin{cases} x' = 16 - xy, \\ y' = x - y^3. \end{cases}$$

(a) Find all the critical points for the system.

Solution. The critical points satisfy

$$\begin{cases} 16 - xy = 0, \\ x - y^3 = 0, \end{cases}$$

which gives $(8, 2)$ and $(-8, -2)$.

(b) Using the theory of stability for almost linear systems, discuss the stability of each critical point, and determine their nature (saddle point, spiral, node, etc), when possible.

Solution. First we analyze $(8, 2)$. Set $u = x - 8$, $v = y - 2$, so that the system becomes

$$\begin{cases} u' = 16 - (u+8)(v+2), \\ y' = (u+8) - (v+2)^3, \end{cases}$$

or yet

$$\begin{cases} u' = -2u - 8v - uv, \\ y' = u - 12v - 6v^2 - v^3. \end{cases}$$

Since the non-linear terms are polynomials of degree at least two in u, v, this system is almost linear. We find the eigenvalues $\lambda = -7 \pm \sqrt{17}$ and conclude that $(8, 2)$ is an asymptotically stable improper node.

Next, we analyze $(-8, -2)$. Set $u = x + 8$, $v = y + 2$, so that the system becomes, after simplifying

$$\begin{cases} u' = 2u + 8v - uv, \\ y' = u - 12v + 6v^2 - v^3. \end{cases}$$

Again the non-linear terms are polynomials of degree at least two in u, v, and the system is almost linear. We find the eigenvalues $\lambda = -5 \pm \sqrt{57}$ and conclude that $(-8, -2)$ is a saddle point.

(c) Sketch the phase plane diagram.

Solution. We compute the eigenvectors associated with the linear part of the u, v system above. For $(8, 2)$ we we find $(5 - \sqrt{17}, 1) \approx (1, 1)$ for $\lambda = -7 - \sqrt{17}$, and $(5 + \sqrt{17}, 1) \approx (9, 1)$ for $\lambda = -7 + \sqrt{17}$. For $(-8, -2)$ we find $(7 - \sqrt{57}, 1) \approx (-1, 1)$ for $\lambda = -5 - \sqrt{57}$, and $(7 + \sqrt{57}, 1) \approx (15, 1)$ for $\lambda = -5 + \sqrt{57}$.

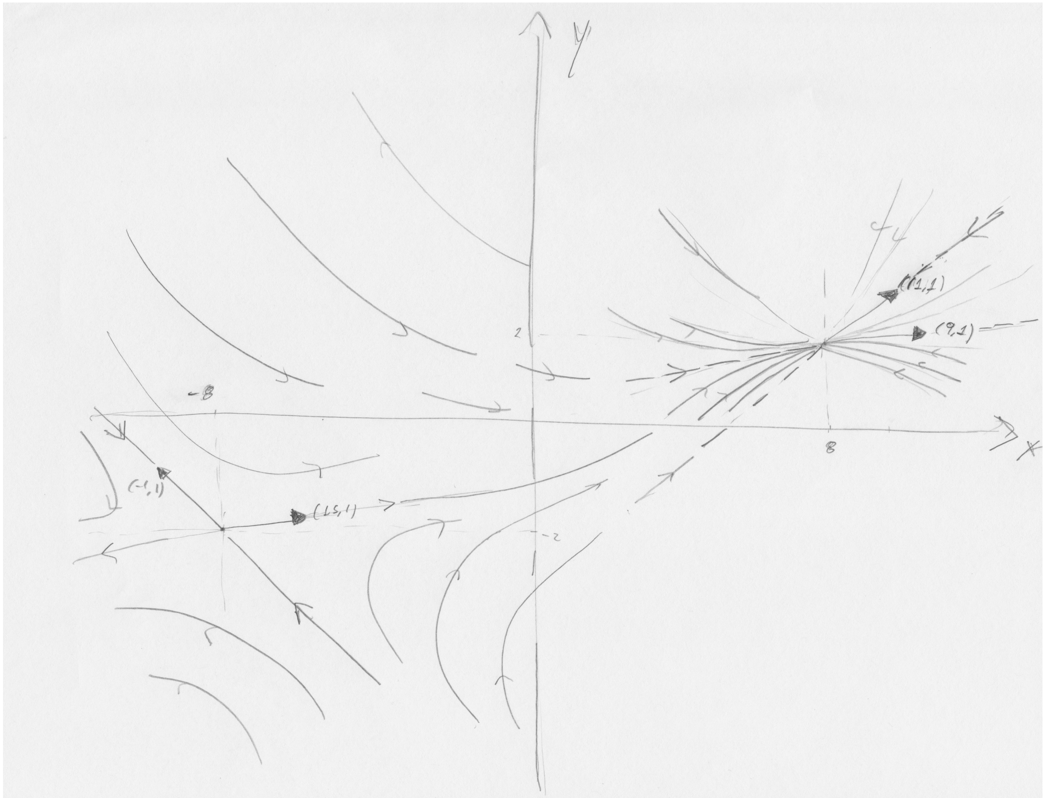

FIGURE 3. Phase diagram sketch.

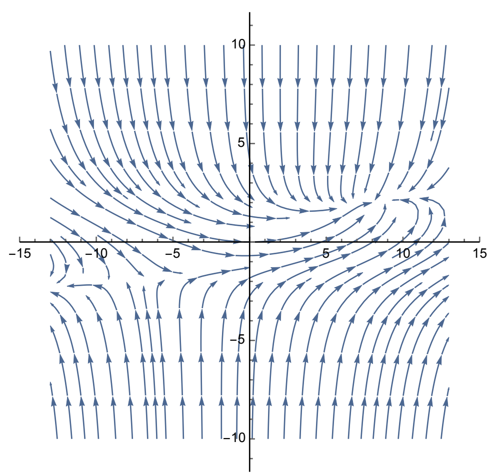

FIGURE 4. Computer generated phase diagram.

Question 7. Consider the system

$$\begin{cases} x' = -y - x^3 - xy^2, \\ y' = x - x^2y - y^3. \end{cases}$$

Discuss the stability of $(0,0)$. *Hint:* Use a Lyapunov function based on the energy of the associated linear system.

Solution. The linear system

$$\begin{cases} x' = -y, \\ y' = x \end{cases}$$

has eigenvalues $\lambda = \pm i$. Therefore, we cannot apply the theory of almost linear systems. Set $V(x,y) = ax^2 + by^2$, where a and b are constants to be determined. Computing

$$\begin{aligned} \frac{d}{dt}V(x,y) &= 2axx' + 2byy' \\ &= 2ax(-y - x^3 - xy^2) + 2by(x - x^2y - y^3) \\ &= (-2a + 2b)xy - 2ax^4 - 2ax^2y^2 - 2bx^2y^2 - 2by^4. \end{aligned}$$

As we want $\dot{V}(x,y)$ to have a definite sign, we choose $a = b$ to kill the term in xy. Also, if $a > 0$ then $\dot{V}(x,y) < 0$, and if we choose $a = 1$:

$$\begin{aligned} &= -2x^4 - 4x^2y^2 - 2y^4 \\ &= -2(x^2 + y^2)^2. \end{aligned}$$

Since in a neighborhood of the origin V is positive definite and $\dot{V}$ is negative definite, Lyapunov's stability theorem implies that $(0,0)$ is asymptotically stable.

Question 8. Make sure to understand the following:

- How the formulas you memorize/derive for general solutions are adapted for the case of an initial value problem.
- The definitions of chapter 9 (matrix exponential, fundamental matrix, etc).
- The definitions given in chapter 12, particularly those involving stability/instability and the definitions with ε-δ.
- Be prepared to state, and know how to use, the important theorems of chapter 12.
- How to draw pictures illustrating both the ideas involving stability, and the formal statements of the relevant theorems of chapter 12.
- How to use Lyapunov functions (the examples given in class can be useful here).