VANDERBILT UNIVERSITY
MATH 208 — ORDINARY DIFFERENTIAL EQUATIONS
PRACTICE FINAL.

The following directions must be followed in the Final Exam. Keep them in mind while preparing for the test.

Instructions. Show all work on your exam paper. Little or no credit may be awarded—even if the final answer is correct—if a full explanation is not provided. Write legibly. Make statements (for instance, write equal signs if you think two expressions are equal.)

Question 1. Find the general solution of the differential equations below.

(a) $x'' - x' - 6x = 0$.

(b) $x'' + 8x' + 16x = 0$.

Question 2. Consider the initial value problem.

$$\frac{dx}{dt} + t\cos x = 0,\ x(1) = \pi. \tag{1}$$

(a) What does the existence and uniqueness theorems for differential equations say about solvability of (1)? Notice that we have learned more than one existence and uniqueness theorem. Apply whichever one you find appropriate and, in doing so,

- State and check the assumptions of the theorem you are using.
- Make precise statements, and state where the solution is defined.

(b) Solve the initial value problem (1).

Question 3. Given an $n \times n$ matrix A:

(a) What is the definition of a fundamental matrix for $x' = Ax$? If $X(t)$ is a fundamental matrix, what is its relation to e^{At}?

(b) What is the definition of a generalized eigenvector of A?

(c) Find a formula for the general solution of the system

$$x'(t) = Ax(t) + f(t).$$

Question 4. Consider the matrix

$$A = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}.$$

The characteristic polynomial of A is given by $p(\lambda) = -\lambda(\lambda - 5)^2$ and

$$(A - 5I)^2 = \begin{bmatrix} -4 & 20 & -8 \\ -5 & 25 & -10 \\ 2 & -10 & 4 \end{bmatrix}.$$

Using generalized eigenvectors, find a fundamental matrix $X(t)$ for the system

$$x' = Ax.$$

Question 5. Consider the system

$$\begin{cases} x' = 2x + y + 9, \\ y' = -5x - 2y - 22. \end{cases}$$

(a) Determine the nature (saddle point, spiral, node, etc) and stability (stable, asymptotic stable, etc) of the critical points.

(b) Compute the eigenvectors.

(c) Sketch the phase plane diagram.

Question 6. Consider the system

$$\begin{cases} x' = 16 - xy, \\ y' = x - y^3. \end{cases}$$

(a) Find all the critical points for the system.

(b) Using the theory of stability for almost linear systems, discuss the stability of each critical point, and determine their nature (saddle point, spiral, node, etc), when possible.

(c) Sketch the phase plane diagram.

Question 7. Consider the system

$$\begin{cases} x' = -y - x^3 - xy^2, \\ y' = x - x^2 y - y^3. \end{cases}$$

Discuss the stability of $(0,0)$. *Hint:* Use a Lyapunov function based on the energy of the associated linear system.

Question 8. Make sure to understand the following:

- How the formulas you memorize/derive for general solutions are adapted for the case of an initial value problem.
- The definitions of chapter 9 (matrix exponential, fundamental matrix, etc).
- The definitions given in chapter 12, particularly those involving stability/instability and the definitions with ε-δ.
- Be prepared to state, and know how to use, the important theorems of chapter 12.
- How to draw pictures illustrating both the ideas involving stability, and the formal statements of the relevant theorems of chapter 12.
- How to use Lyapunov functions (the examples given in class can be useful here).