VANDERBILT UNIVERSITY
MATH 208 — ORDINARY DIFFERENTIAL EQUATIONS
EXAMPLES OF SECTION 12.3.

Question 1. Show that the given system is almost linear near the origin and discuss the type and stability of the critical point at the origin.

$$\begin{cases} x' = -2x + 2xy, \\ y' = x - y + x^2. \end{cases}$$

Question 2. Do the same as in question 1 for the system

$$\begin{cases} x' = \sin(y - 3x), \\ y' = \cos(x) - e^y. \end{cases}$$

Solutions.

1. The system has the form

$$\begin{cases} x' = ax + by + F(x,y), \\ y' = cx + dy + G(x,y), \end{cases}$$

with $a = -2$, $b = 0$, $c = 1$, $d = -1$, $F(x,y) = 2xy$, and $G(x,y) = x^2$. Thus, $ad - bc \neq 0$ (recall that, by definition, an almost linear system must satisfy $ad - bc \neq 0$). We need to check

$$\frac{F(x,y)}{\| (x,y) \|} = \frac{F(x,y)}{\sqrt{x^2 + y^2}} \to 0,$$

and

$$\frac{G(x,y)}{\| (x,y) \|} = \frac{G(x,y)}{\sqrt{x^2 + y^2}} \to 0,$$

as $\| (x,y) \| \to 0$. For $x > 0$ and $y > 0$ we have

$$\begin{aligned} \frac{F(x,y)}{\sqrt{x^2+y^2}} &= \frac{2xy}{\sqrt{x^2+y^2}} \\ &= \frac{2xy}{\sqrt{x^2+y^2}} \\ &= \frac{2}{\sqrt{x^2+y^2}/\sqrt{x^2y^2}} \\ &= \frac{2}{\sqrt{\frac{1}{y^2}+\frac{1}{x^2}}} \end{aligned}$$

so that $F(x,y) \to 0$ when $\| (x,y) \| \to 0$, with a similar argument when x, y, or both, are negative. For $G(x,y)$, notice that

$$0 \leq \frac{G(x,y)}{\sqrt{x^2+y^2}} = \frac{x^2}{\sqrt{x^2+y^2}} \leq \frac{x^2}{\sqrt{x^2}}.$$

so $G(x,y) \to 0$ when $\| (x,y) \| \to 0$ by the squeeze theorem. Therefore the system is almost linear.

The eigenvalues of the associated linear system are -1 and -2, and we conclude that $(0,0)$ is an asymptotically stable improper node.

2. At first sight, it seems that the system cannot be almost linear, since it has no linear terms, hence $a = b = c = d = 0$ and $ad - bc = 0$. However, the system can be rewritten as an almost linear system as follows. Since sin, cos, and the exponential have Taylor expansions that converge for all values, we can write

$$\sin(y-3x) = y - \frac{y^3}{6} + O(y^5) + \Big(-3 + \frac{3y^2}{2} - \frac{y^4}{8} + O(y^5)\Big)x + \Big(-\frac{9y}{2} + \frac{3y^3}{4} + O(y^5)\Big)x^2 + \Big(\frac{9}{2} - \frac{9y^2}{2} + \frac{3y^4}{16} + O(y^5)\Big)x^3 + \Big(\frac{27y}{8} - \frac{9y^3}{16} + O(y^5)\Big)x^4 + O(x^5),$$

where $O(y^5)$ means terms involving powers of y of order at least five, and similarly for $O(x^5)$. The above can be written as

$$\sin(y-3x) = y - 3x + F(x,y),$$

where $F(x,y)$ involves powers of xy of the form $x^m y^n$ with $m + n \geq 3$. This implies

$$\frac{F(x,y)}{\| (x,y) \|} \to 0,$$

as $\| (x,y) \| \to 0$.

Similarly,

$$\cos(x) - e^y = -y + G(x,y),$$

with

$$\frac{G(x,y)}{\| (x,y) \|} \to 0,$$

as $\| (x,y) \| \to 0$.

The system now has $a = -3$, $b = 1$, $c = 0$, $d = -1$, and is almost linear. The eigenvalues of the associated linear system are -3 and -1, and we conclude that $(0,0)$ is an asymptotically stable improper node.

URL: `http://www.disconzi.net/Teaching/MAT208-Fall-14/MAT208-Fall-14.html`