VANDERBILT UNIVERSITY — MATH 208: DIRECTION FIELDS AND RELATED TOPICS

Contents

1. Existence and uniqueness of solutions.

We have seen in class simple examples of how to solve a differential equation (D.E.). We have also seen that, while finding a solution to D.E. may be a hard task, it is relatively easy to *verify* whether a given function is a solution. Similar statements hold for an initial value problem (I.V.P).

From a more general point of view, a natural question is whether there always exists a solution to a given D.E. or I.V.P. We can also ask if a solution, once known to exist, is unique. Notice that uniqueness is a non-trivial question only for an I.V.P. in that, as saw in class, a general D.E. will typically have infinitely many solutions due to the presence of arbitrary constants.

Consider the initial value problem:

$$\begin{cases} F(t, x(t), x'(t), \dots, x^{(n)}(t)) = 0, \\ x(t_0) = x_0, \\ x(t_0) = x_1, \\ \qquad \vdots \\ x^{(n-1)}(t_0) = x_{n-1}. \end{cases} \tag{1.1}$$

The two fundamental questions about (1.1) are:

I. Does (1.1) have a solution? This is the problem of *existence* of solutions.

II. If one knows that (1.1) has a solution, is such a solution unique? This is the problem of *uniqueness* of solutions.

Question I can be rephrased as follows. Given F, and n numbers $x_0, \dots, x_{n-1}$, does there exist a function x_{sol}, defined on a neighborhood of the point $t_0 \in \mathbb{R}$, such that $F(t, x_{\text{sol}}(t), x'_{\text{sol}}(t), \dots, x^{(n)}_{\text{sol}}(t)) = 0$, and the values of x_{sol}, and its derivatives up to order $n-1$, evaluated at t_0, satisfy $x_{\text{sol}}(t_0) = x_0, \dots, x^{(n-1)}_{\text{sol}}(t_0) = x_{n-1}$? Question II can be rephrased as follows. Suppose that x_{sol} and y_{sol} are two functions that answer question I affirmatively, i.e., x_{sol} and y_{sol} are both solutions of the I.V.P. (1.1). Is it the case that $x_{\text{sol}} = y_{\text{sol}}$?

Some students may be puzzled by these questions, as one may initially think that they are trivial. "How can an I.V.P not have a solution?" and "How can we have two, or more, different solutions to a problem?" he or she might ask. But if we think a little, we quickly realize that even for basic algebraic equations it is not always the case that we can find solutions or that solutions are unique. For instance, the equation for the variable z, $z^2 + 1 = 0$, does not have a solution over the real numbers (it will have a solution if we allow complex numbers, but if we insist that solutions ought to be real, then the equation has no solution). And $z^2 - 3z + 2 = 0$ has two different solutions, namely, $z = 1$ and $z = 2$. Thus, we should not be surprised that, in the more sophisticated setting of D.E., where solutions are functions, there will be cases where solutions do not exist or are not unique. In fact, the D.E. $(x')^2 + 1 = 0$ has no (real) solution. As another example, as an exercise, students are encouraged to show that the I.V.P.

$$\begin{cases} xx' = 4t, \\ x(1) = 0, \end{cases}$$

has no solution (see Example 5 on page 8 of the textbook). Another example is given further below.

It is useful to have a criterion that guarantees that a given I.V.P. has a solution, and that such a solution is unique. For first order equations, the following theorem is available (page 11 of the textbook).

Theorem 1.1. *Consider the following I.V.P.*

$$\begin{cases} x' = f(t, x), \\ x(t_0) = x_0. \end{cases} \tag{1.2}$$

If f and $\partial_x f$ are continuous functions on some rectangle

$$R = \{(t, x) : a < t < b, c < x < d\},$$

containing the point (t_0, x_0), then (1.2) has a unique solution x_{sol} defined on some interval $(t_0 - \delta, t_0 + \delta)$, where δ is some positive number.

As an example, consider

$$\begin{cases} x' = \sin t - \cos x, \\ x(\pi) = 0. \end{cases}$$

In this case, $f(t, x) = \sin t - \cos x$, so that $\partial_x f(t, x) = \sin x$. Since both sin and cos are continuous functions, we conclude that theorem 1.1 can be applied, and therefore the problem has a solution, which is unique, defined on some interval about $t = \pi$.

Notice that theorem 1.1 is "abstract," in the sense that it does not give any idea of how the solution looks like, nor does it say how such a solution can be explicitly found. In the above example, even though we know that there exists a unique function such that $x' = \sin t - \cos x$ and $x(\pi) = 0$, the form of this function remains unknown.

It is important to stress that theorem 1.1 does *not* say that a solution does not exist when its hypotheses are not fulfilled. For instance, take

$$\begin{cases} x' = 2\sqrt{|x|}, \\ x(0) = 0. \end{cases} \tag{1.3}$$

In this case, $f(t, x) = 2\sqrt{|x|}$, which is continuous, but

$$\partial_x f(t, x) = \begin{cases} \frac{1}{\sqrt{x}}, & \text{if } x > 0, \\ -\frac{1}{\sqrt{-x}}, & \text{if } x < 0. \end{cases}$$

is not continuous at $x = 0$. Thus, theorem 1.1 cannot be applied, but this does not mean that the problem does not have a solution. In fact, for any $a \geq 0$, the function

$$x_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ (t - a)^2, & \text{if } t \geq a. \end{cases}$$

is a solution to the I.V.P. (1.3). Notice, however, that such a solution is not unique, since we can pick any $a \geq 0$.

2. Direction fields.

From the previous section, we see that there are instances where we can ensure that a given I.V.P. has a unique solution, but an explicit formula for the solution may not be available. As another example, consider

$$\begin{cases} x' = e^{-t^2}, \\ x(0) = 1. \end{cases}$$

Theorem 1.1 can be applied to guarantee the existence and uniqueness of a solution. Furthermore, we can write a formula for the solution, namely,

$$x(t) = 1 + \int_0^t e^{-s^2}\, ds.$$

Unfortunately, as students probably learned in calculus, it is impossible to compute the integral $\int_0^t e^{-s^2}\, ds$ in terms of elementary functions. In light of situations like this, it is important to develop a method that allows us to grasp the behavior of solutions when an explicit formula is not available. One such tool is based on *direction fields,* which we now explain.

Consider $x' = \cos(xt)$. Students can check that theorem 1.1 can always be applied, regardless of the initial condition. Hence, we know that given (t_0, x_0), there exists a unique solution with initial condition $x(t_0) = x_0$. How do solutions look like?

To answer that question, let us recall the geometric interpretation of the derivative: it is the slope of the tangent line to the graph. Thus, $x' = \cos(tx)$ mans that, at the point (t, x), the solution has slope equal to $\cos(tx)$. As a consequence, although we do not know an expression for the curve $x(t)$ which is the solution passing through a given point, we do have a formula for its slope. This is enough to determine the graph of solutions, as follows. Construct a table with the values of t, x, and $x' = \cos(tx)$.

t-value	x-value	slope $x' = \cos(tx)$
0.50	0.50	0.968912
0.75	0.50	0.930508
1.00	0.50	0.877583
0.50	0.75	0.930508
0.75	0.75	0.845924
1.00	0.75	0.731689
0.50	1.00	0.877583
0.75	1.00	0.731689
1.00	1.00	0.540302

Next, at each point (t, x) on the tx-plane, we draw a vector whose slope is the value x' computed in the third column. *By construction, solutions have to always be tangent to these vectors.* If we use enough points, the result is shown in figure 1. The plot in figure 1 is called a *direction field.* From the direction field, we can now read off the behavior of solutions. For example, suppose we are interested in the solution with initial condition $x(0) = 0$. The graph of this solution has to cross the origin, and from the picture we see that its slope is 45° (in fact, for $t = 0 = x$, $x' = \cos(0) = 1$). Since the curve $x(t)$ has to be tangent to the vectors we drew, it is seen that, as t increases, $x(t)$ also increases, but with the vectors representing x' "rotating" clock-wise, until a point at approximately

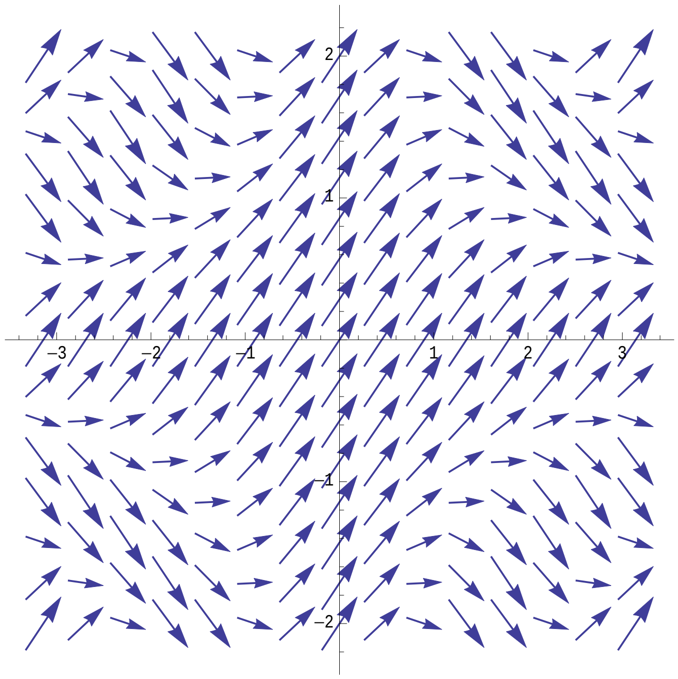

FIGURE 1. Direction field for $x' = \cos(tx)$.

$t = 1.5$ when the slope becomes horizontal. Then, the curve starts descending toward the t-axis. Drawing the curve in detail yields the plot in figure 2.

Naturally, figures 1 and 2 are computer generated. Some of the homework problems will require the use of computers to build similar plots. Here, we have used the software Mathematica. Students can use Mathematica or any other application of their choice. Those not interested in purchasing one of these tools may find many free applications for plotting direction fields on the web. One package, which is suggested on the textbook, is available at:

http://alamos.math.arizona.edu/~rychlik/index.php
under "Software."

3. The Euler method.

The use of direction fields, illustrated in the previous section, provides a way of analyzing solutions to a D.E. in the absence of an explicit formula. While this is useful, in many real life applications one needs a more precise and quantitative assessment of solutions to a given problem. The *Euler*[1] *method,* which we shall now present, fulfills this requirement, providing a systematic way of constructing an explicit *approximation* to the solution of an I.V.P.

[1]From Wikipedia: "Leonhard Euler (15 April 1707 - 18 September 1783) was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function (...) Euler is considered to be the pre-eminent mathematician of the 18th century and one of the greatest mathematicians to have ever lived. He is also one of the most prolific mathematicians; his collected works fill 60-80 quarto volumes."

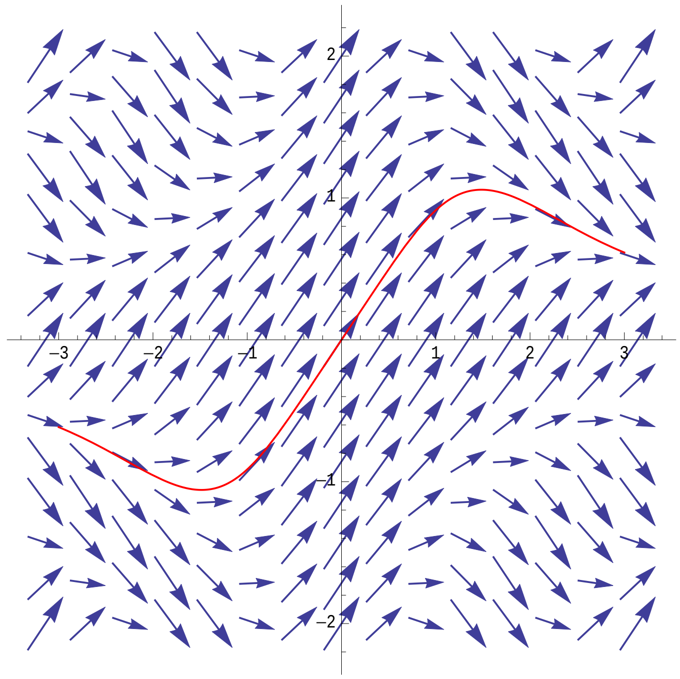

FIGURE 2. Direction field for $x' = \cos(tx)$, along with the solution with initial condition $x(0) = 0$.

The basic idea of the Euler method is, once again, to rely on the geometric meaning of the derivative. Suppose we want to solve

$$\begin{cases} x' = f(t, x), \\ x(t_0) = x_0. \end{cases}$$

Assume that theorem 1.1 can be applied, so that we know that a solution satisfying the given initial condition exists and is unique. Denote the solution by x_{sol}. Since t_0, $x_{\text{sol}}(t_0) = x_0$, and f are given, we can compute the slope of the tangent line to x_{sol} at the point (t_0, x_0), namely, $x'_{\text{sol}}(t_0) = f(t_0, x_0)$. From this information, we would like to determine the value of the solution x_{sol} at another *chosen* point t_1 different than t_0. If t_1 is very close to t_0 then, from basic calculus, we know that the graph of $x_{\text{sol}}(t)$ is very close to the graph of the tangent line to the point (t_0, x_0). But the latter can be found; it is given by

$$x = x_0 + (t - t_0) f(t_0, x_0). \tag{3.1}$$

Thus, the value we are seeking to determine, $x_{\text{sol}}(t_1)$, which we call x_1, can be approximated with the help of (3.1), and we find

$$x_1 = x_{\text{sol}}(t_1) \approx x_0 + (t_1 - t_0) f(t_0, x_0).$$

Now that we know the (approximate) value of $x_{\text{sol}}(t_1)$, we can compute the value of the slope at the point (t_1, x_1), namely, $f(t_1, x_1)$. The graph of x_{sol} in neighborhood of (t_1, x_1) is close to the graph of the tangent line at that point, given by

$$x = x_1 + (t - t_1) f(t_1, x_1).$$

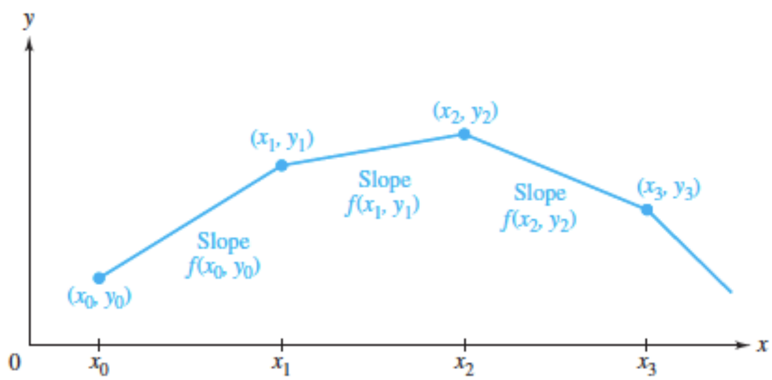

FIGURE 3. Illustration of the Euler method. This is figure 1.15 from page 24 of the textbook.

The reader has probably realized that we can now repeat the argument: if we want the value $x_{\text{sol}}(t_2)$ at a point t_2 which is close to t_1, we can use the approximation

$$x_2 = x_{\text{sol}}(t_2) \approx x_1 + (t_2 - t_1) f(t_1, x_1).$$

Proceeding in this way, one finds a sequence of points

$$\{(t_0, x_0), (t_1, x_1), (t_2, x_2), \dots\}$$

that provides an approximation to the solution x_{sol}. In other words, plotting the points (t_n, x_n), $n = 0, 1, 2, \dots$, an joining them by a straight line, provides an approximation to the graph of x_{sol}. This idea is illustrated in figure 3, except that in this figure the variables have been labeled (x, y) instead of (t, x).

It should be emphasized that this procedure relies crucially on the fact that the points $t_1, t_2, \dots$ are always close to their predecessors, i.e. the difference $t_{n+1} - t_n$, $n = 0, 1, 2, \dots$ has to be very small. The difference $t_{n+1} - t_n$ is called the *step size,* henceforth denoted by h:

$$h = t_{n+1} - t_n.$$

h is a value that is conveniently chosen. Naturally, the smaller the h, the better the approximation. On the other hand, the smaller the h, the larger the number of steps that have to be computed to go from t_0 to a desired value t. Once h is chosen, we have the following recursive formula for t_n:

$$t_{n+1} = t_n + h,$$

or, equivalently,

$$t_{n+1} = (n + 1)h + t_0.$$

Summarizing, the Euler method is encoded in the following recursive relations,

$$\begin{cases} t_{n+1} = t_n + h, \\ x_{n+1} = x_n + h f(t_n, x_n), \end{cases} \tag{3.2}$$

where t_0, x_0, and f are given, and h is chosen.

As an illustration, consider again $x' = \cos(tx)$ with initial condition $x(0) = 0$. Suppose we want to find $x(1.5)$. Choosing $h = 0.5$ and applying (3.2), we find the following sequence of values $\{(t_n, x_n)\}$:

$$\{(0, 0), (0.5, 0.498814), (1., 0.939901), (1.5, 1.10945)\}.$$

The desired approximation, $x(1.5)$, is the last entry in the list, $x(1.5) \approx 1.10945$.

Using a better step size, $h = 0.1$, we find

$$\{(0,0),(0.1,0.1),(0.2,0.199995),(0.3,0.299915),(0.4,0.399511),(0.5,0.498236),(0.6,0.595149),$$
$$(0.7,0.688841),(0.8,0.777439),(0.9,0.858714),(1.,0.930307),(1.1,0.990065),$$
$$(1.2,1.0364),(1.3,1.06853),(1.4,1.0866),(1.5,1.09155)\}.$$

Thus, with $h = 0.1$, $x(1.5) \approx 1.09155$. If we insist in an even smaller h, $h = 0.01$, the list is

$$\{(0.,0.),(0.01,0.01),(0.02,0.02),(0.03,0.03),(0.04,0.04),$$
$$(0.05,0.05),(0.06,0.06),(0.07,0.0699999),$$
$$\dots,$$
$$(1.46,1.06089),(1.47,1.06111),(1.48,1.06122),(1.49,1.06122),(1.5,1.06111)\},$$

and in this case $x(1.5) \approx 1.06111$. Notice how the values of $x(1.5)$ change with different choices of h. Since smaller h's provide better approximations, we see that the last value, $x(1.5) \approx 1.06111$ is a more accurate approximation of the actual value of $x(1.5)$ (which we do not know) than the previous two estimates, i.e., 1.09155 and 1.10945.

But how small h has to be? If the value we want changes every time we change h, how can we be any confident that what we obtained is a good approximation to the actual solution? This is a very important question, whose answer is beyond the scope of our course[2]. Here, it suffices to say that, in many practical situations, one keeps decreasing the value of h until the point where our solution "stabilizes," i.e., the values we compute no longer change very much with decreasing h. For instance, if we go ahead and use $h = 0.001$, we find again $x(1.5) \approx 1.06111$, i.e., the difference to the case $h = 0.01$ happens only after the fifth decimal place. Thus, if we are interested in, say, solutions up to three or four decimal digits, $h = 0.01$ is enough.

We remark that the above values were obtained with the help of a computer. Here, once more, we have employed Mathematica, but students are free to use whatever software they find appealing (again, there are many free tools on the web). In fact, for the homework problems, students will not be asked to use many steps, and the computations, albeit tedious, can be done by hand with the help of a four-function calculator.

URL: `http://www.disconzi.net/Teaching/MAT208-Fall-14/MAT208-Fall-14.html`

[2]The student interested in these type of questions should consider taking a course in numerical analysis.