VANDERBILT UNIVERSITY
MATH 198 —METHODS OF ORDINARY DIFFERENTIAL EQUATIONS
EXAMPLES OF SECTION 8.6.

Question. Solve

$$x^2y'' + xy' + (x^2 - 4)y = 0.$$

Solution. Write the equation as

$$y'' + \frac{1}{x}y' + \frac{x^2 - 4}{x^2}y = 0.$$

Then $x = 0$ is a singular point, since

$$p(x) = \frac{1}{x},$$

and

$$q(x) = \frac{x^2 - 4}{x^2}.$$

This singular point is a regular singular point because

$$xp(x) = 1,$$

and

$$x^2q(x) = x^2 - 4,$$

which are polynomials, thus analytic functions.

Computing

$$p_0 = \lim_{x\to 0} xp(x) = 1,$$

and

$$q_0 = \lim_{x\to 0} x^2q(x) = -4,$$

we find the indicial equation to be

$$r(r-1) + p_0r + q_0 = r^2 - 4 = 0.$$

The solutions of the indicial equation are $r = 2$ and $r = -2$. Although we haven't seen this yet, we'll learn next class that, when there are two distinct roots, we ought to take the larger one (recall that the example done in class had repeated roots, so this issue didn't arise). Thus, we put $r = 2$, and look for a solutions of the form

$$y = x^2\sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^{n+2}.$$

Differentiating and plugging into the equation yields

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_nx^{n+2} + \sum_{n=0}^{\infty}(n+2)a_nx^{n+2} + \sum_{n=0}^{\infty} a_nx^{n+4} - 4\sum_{n=0}^{\infty} a_nx^{n+2} = 0.$$

After some algebra, this becomes

$$5a_1x^3 + \sum_{n=2}^{\infty}((n^2+4n)a_n + a_{n-2})x^{n+2} = 0.$$

This gives

$$a_1 = 0,$$

and

$$a_n = -\frac{1}{n(n+4)}a_{n-2}.$$

Then,

$$a_2 = -\frac{1}{2(6)}a_0 = -\frac{1}{2^1(3!)}a_0,$$

$$a_3 = 0,$$

$$a_4 = -\frac{1}{4(8)}a_2 = \frac{1}{2^3(2!)(4!)}a_0,$$

$$a_5 = 0,$$

$$a_6 = -\frac{1}{6(10)}a_4 = -\frac{1}{2^5(3!)(5!)}a_0,$$

$$a_7 = 0,$$

and we see that

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n-1}n!(n+2)!},$$

and

$$a_{2n+1} = 0.$$

We find as solution

$$y = x^2\sum_{n=0}^{\infty}\frac{(-1)^n a_0}{2^{2n-1}n!(n+2)!}x^{2n}.$$

URL: `http://www.disconzi.net/Teaching/MAT198-Spring-14/MAT198-Spring-14.html`