VANDERBILT UNIVERSITY
MATH 198 —METHODS OF ORDINARY DIFFERENTIAL EQUATIONS
EXAMPLES OF SECTION 7.7.

Question. Show that

$$\mathcal{L}\{f * g\}(s) = F(s)G(s),$$

where f and g are piecewise continuous on $[0, \infty)$ and of exponential order α.

Solution. Write, for $s > \alpha$,

$$\mathcal{L}\{f * g\}(s) = \int_0^\infty e^{-st} \left(\int_0^t f(t-r)g(r)\,dr \right) dt.$$

Since

$$\int_0^t f(t-r)g(r)\,dr = \int_0^\infty u(t-r)f(t-r)g(r)\,dr,$$

where u is the unit step function (so $u(t-r) = 0$ for $r > t$), we have

$$\mathcal{L}\{f * g\}(s) = \int_0^\infty e^{-st} \left(\int_0^\infty u(t-r)f(t-r)g(r)\,dr \right) dt.$$

Because $s > \alpha$, we can change the order of integration,

$$\mathcal{L}\{f * g\}(s) = \int_0^\infty g(r) \left(\int_0^\infty e^{-st}u(t-r)f(t-r)\,dt \right) dr.$$

But

$$\int_0^\infty e^{-st}u(t-r)f(t-r)\,dt = e^{-sr}F(s)$$

by properties of the Laplace transform. Thus,

$$\mathcal{L}\{f * g\}(s) = \int_0^\infty g(r)e^{-sr}F(s)\,dr = F(s) \int_0^\infty e^{-sr}g(r)\,dr = F(s)G(s).$$

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