VANDERBILT UNIVERSITY
MATH 198 —METHODS OF ORDINARY DIFFERENTIAL EQUATIONS
EXAMPLES OF SECTIONS 7.2 AND 7.3.

Question. Suppose that the Laplace transform $\mathcal{L}\{f\}(s) = F(s)$ exists for $s > \alpha$. Show that

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a),$$

for $s > \alpha + a$.

Solution. We use the definition of the Laplace transform to compute

$$\begin{aligned} \mathcal{L}\{e^{at} f(t)\} &= \int_0^\infty e^{-st} e^{at} f(t)\, dt \\ &= \int_0^\infty e^{-(s-a)t} f(t)\, dt \\ &= \int_0^\infty e^{-rt} f(t)\, dt, \end{aligned}$$

where $r = s - a$. The last integral is the Laplace transform of f with independent variable r, i.e.,

$$\int_0^\infty e^{-rt} f(t)\, dt = F(r).$$

Since $r = s - a$, we conclude

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-rt} f(t)\, dt = F(r) = F(s-a).$$

$F(r)$ is defined for $r > \alpha$, thus $\mathcal{L}\{e^{at} f(t)\}$ is defined for $s > \alpha + a$.

URL: `http://www.disconzi.net/Teaching/MAT198-Spring-14/MAT198-Spring-14.html`